Topology, Combinatory and Arithmetic for Line Arrangements in the Projective Plane

Enrique ARTAL BARTOLO

Departamento de Matemáticas
Facultad de Ciencias
Instituto Universitario de Matemáticas y sus Aplicaciones
Universidad de Zaragoza

MATEMATISKA KOLLOKVIET
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Arrangements and Combinatorics

$\mathcal{A}$ central hyperplane arrangement in $\mathbb{K}^{n+1}$: finite family

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$\mathcal{A}$ affine hyperplane arrangement in $\mathbb{K}^{n+1}$: finite family

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$\mathcal{A}$ hyperplane arrangement in $\mathbb{P}^{n+1}(\mathbb{K})$: finite family

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$\mathcal{A}$ line arrangement in $\mathbb{P}^2(\mathbb{K})$: finite family

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Combinatorics

Matroid (*equation-free linear dependence*)

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Intersection pattern

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Definition

Line combinatorics $\mathscr{C} := (\mathscr{L}, \mathscr{P})$, where $\mathscr{P} \subset \mathcal{P}(\mathscr{L})$ such that

- $\forall \ell_1, \ell_2 \in \mathscr{L}$, $\ell_1 \neq \ell_2$, $\exists! p \in \mathscr{P}$ such that $\ell_1, \ell_2 \in p$
- $\forall p \in \mathscr{P}$, $m_p := \# p \geq 2$.

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Realization space of a line combinatorics

- $\mathcal{A}$ line arrangement in $\mathbb{P}^2(\mathbb{K}) \Longrightarrow \mathscr{C}(\mathcal{A})$:

$$(\mathcal{A}, \{\text{multiple points}\}) \leftrightarrow (\mathscr{L}, \mathscr{P})$$

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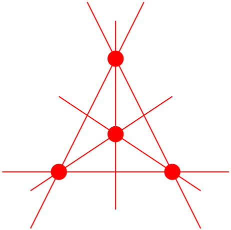

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- $\mathcal{A}_{\mathbb{C}} : (x^3 - y^3)(y^3 - z^3)(z^3 - x^3) = 0.$

Realization and moduli space II

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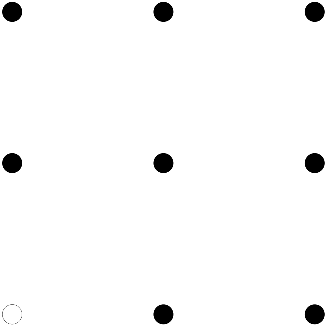
- *McLane combinatorics* $\mathscr{N}$.



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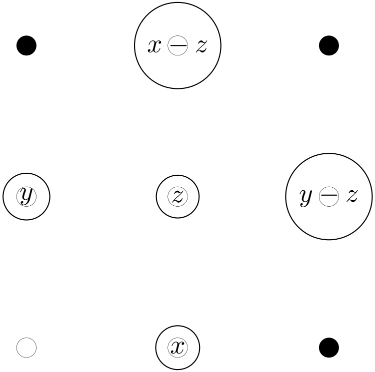

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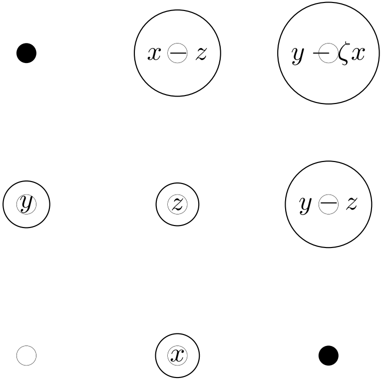

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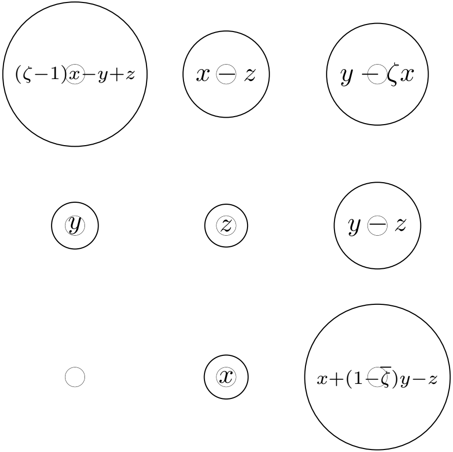

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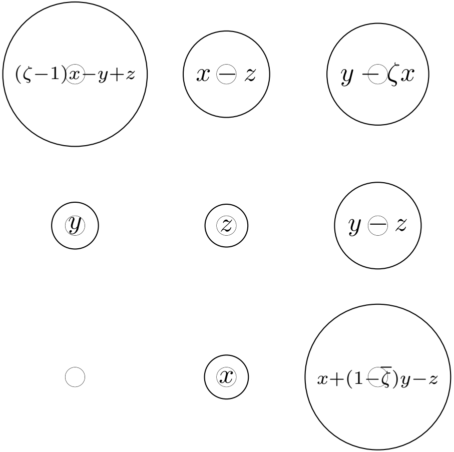

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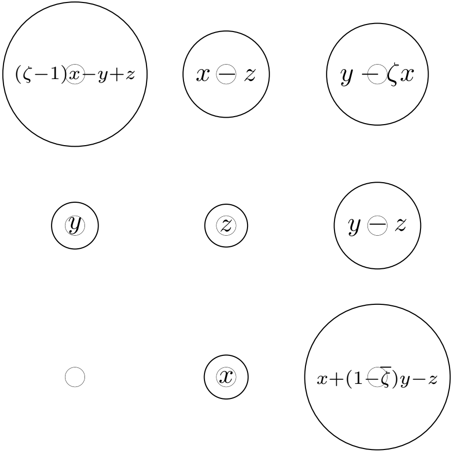

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- $\mathbb{Z}^{\mathscr{L}} = \bigoplus_{\ell \in \mathscr{L}} \mathbb{Z} x_\ell,$

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- $H^0_{\mathscr{C}} = \mathbb{Z}$, $H^j_{\mathscr{C}} \cong H^j(\mathbb{P}^2 \setminus \mathcal{A}; \mathbb{Z})$ dual of $H_j^{\mathscr{C}}$, $j = 1, 2$.

Negative results

Theorem (Rybnikov)

$\nexists \varphi : \pi_1(\mathbb{P}^2 \setminus \mathcal{N}_+) \to \pi_1(\mathbb{P}^2 \setminus \mathcal{N}_-)$ *group automorphism inducing the identity on homology.*

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Guidelines of the proof.

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- *Minimal* solutions in $\frac{1}{3}\mathbb{Z}$.

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$\nexists \rho : (\mathbb{P}^2, \mathcal{N}_+) \to (\mathbb{P}^2, \mathcal{N}_-)$ *homeomorphism respecting orientations and ordering.*

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Order

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2. It does not happen using *truncated Alexander invariant* with the same ideas as in McLane's.

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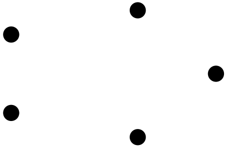
Pentagon



$\mathbb{Q}(\sqrt{5})$

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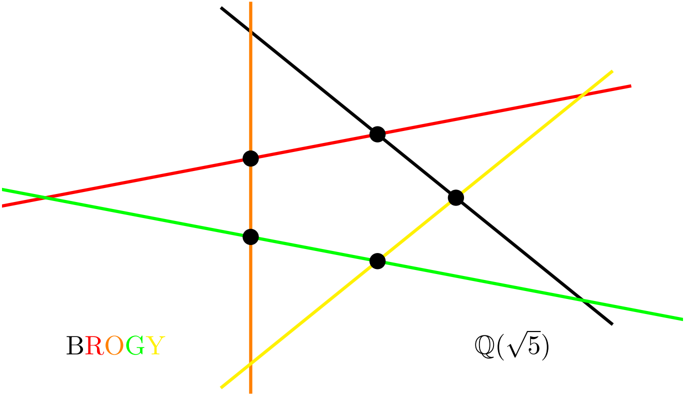
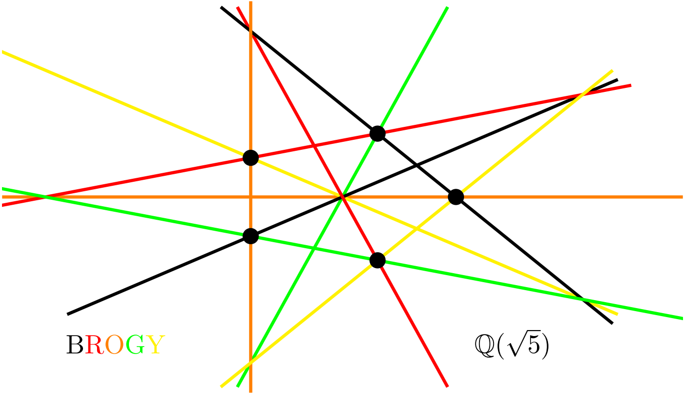


Pentagon



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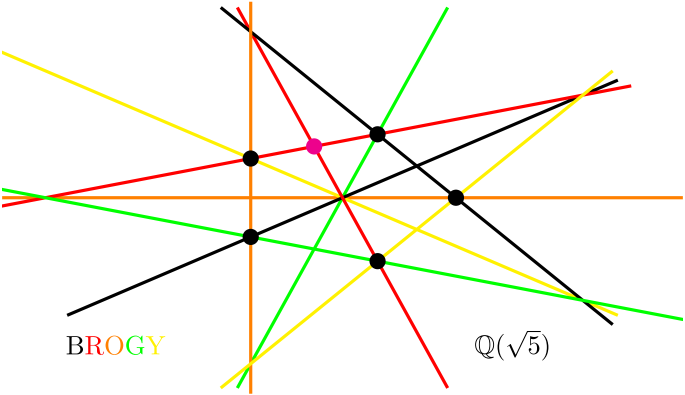

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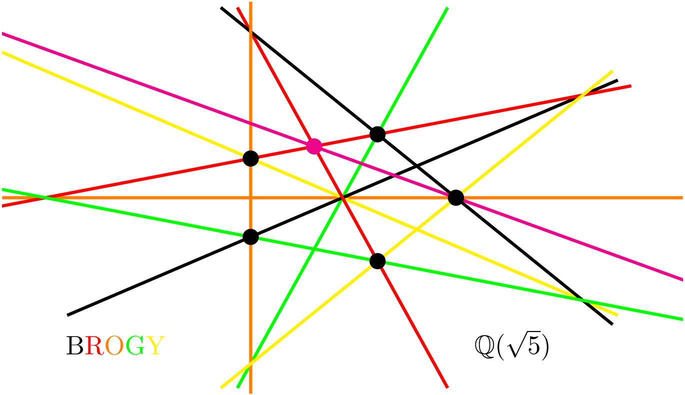
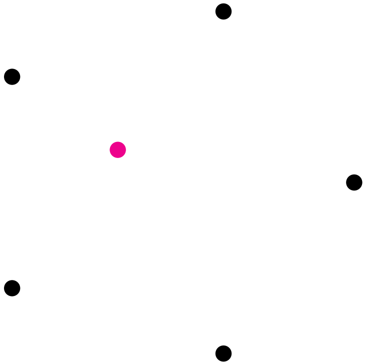


Pentagram



$$\sqrt{5} \mapsto -\sqrt{5}$$

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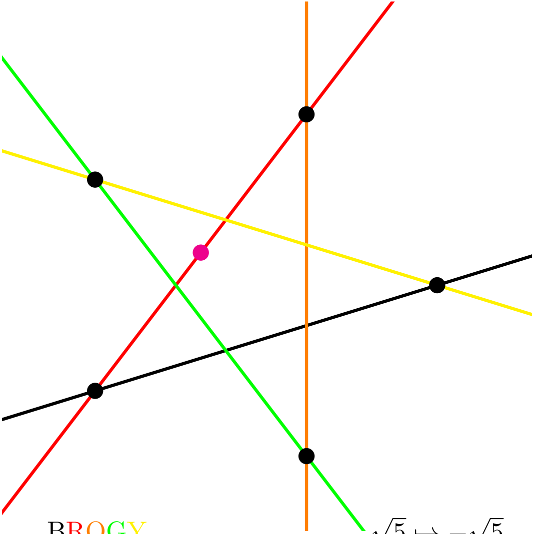

BROGY

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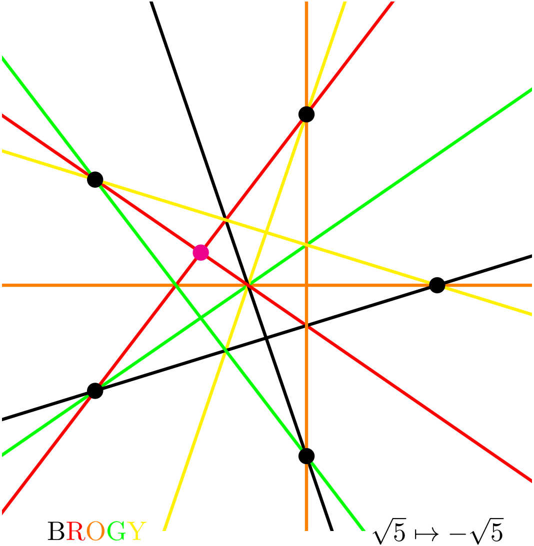

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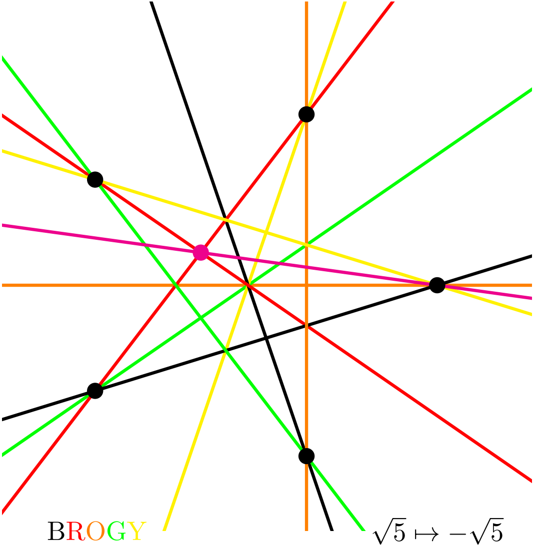
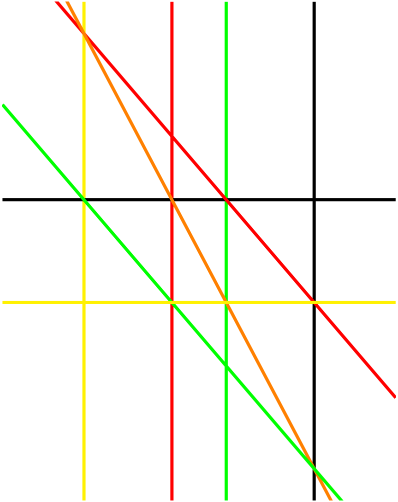


Vertical versions



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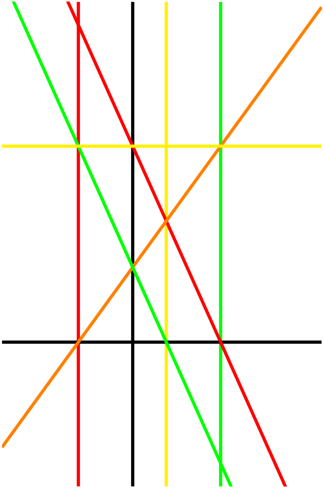

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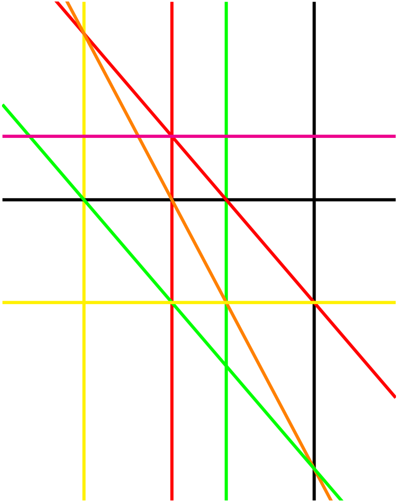

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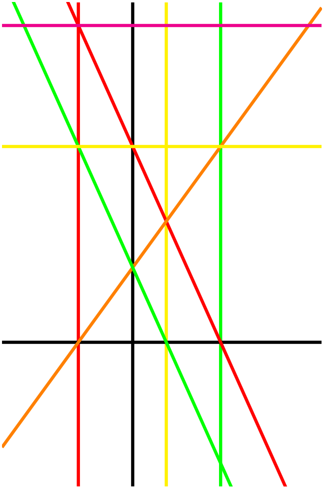

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Braid Monodromy and Topology

Moduli space

$\#\mathscr{M}_{\mathbb{C}}(\mathscr{A}) = 2$

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Theorem (__, Carmona, Cogolludo, Marco)

There is no homeomorphism $\Phi : (\mathbb{P}^2, \mathcal{A}_{\sqrt{5}}) \to (\mathbb{P}^2, \mathcal{A}_{-\sqrt{5}})$.

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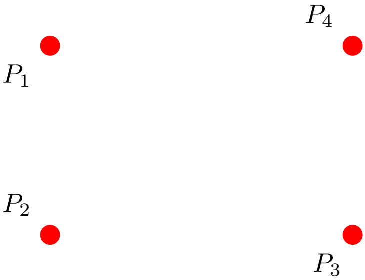
- Finite presentations of $G_{\pm} = \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{\pm\sqrt{5}})$.
- $G_{\pm}$ have the same finite quotients (not known if isomorphic!).

$\mathscr{G}_{91}$ combinatorics



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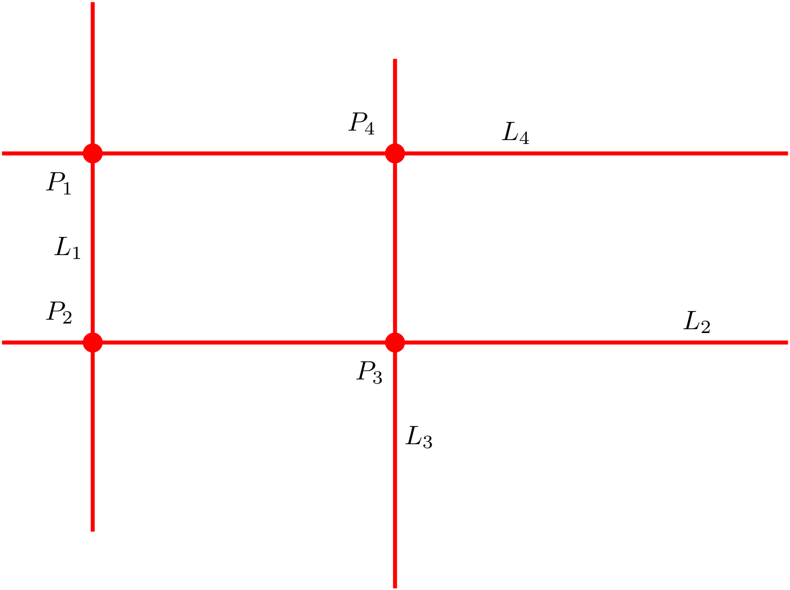

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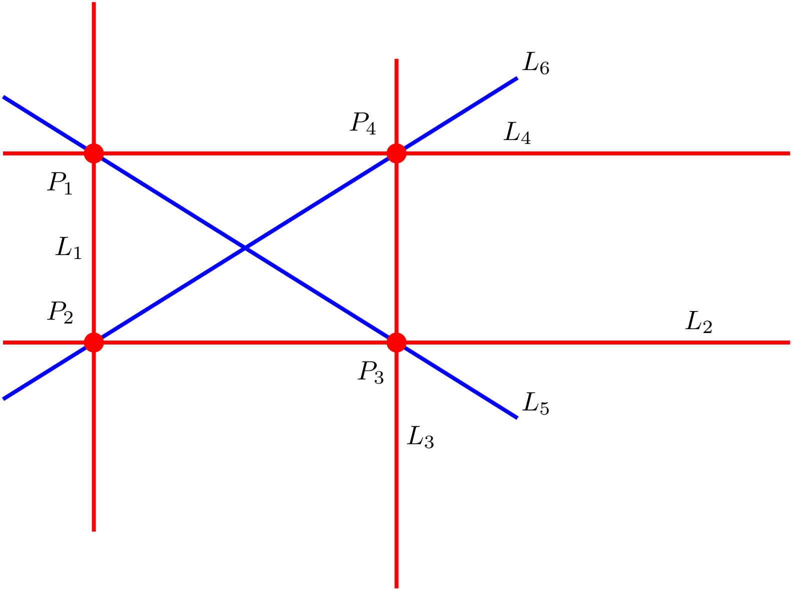

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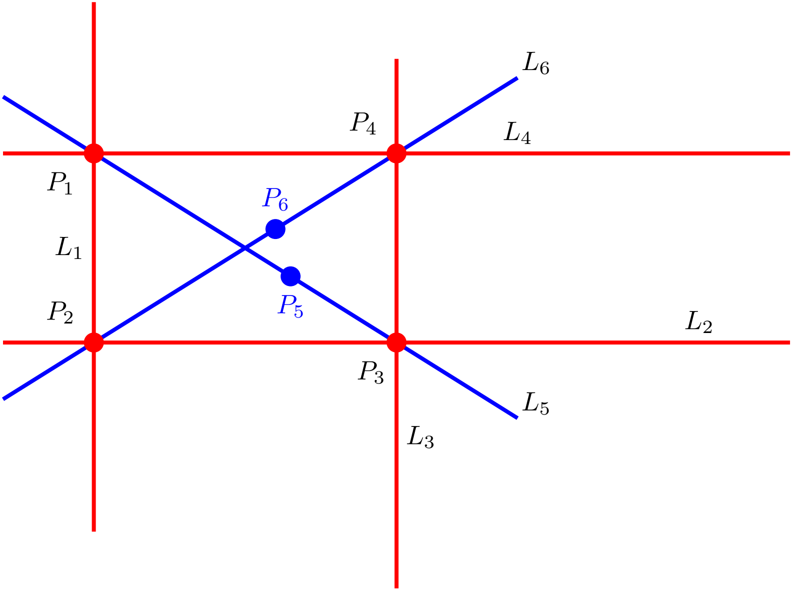

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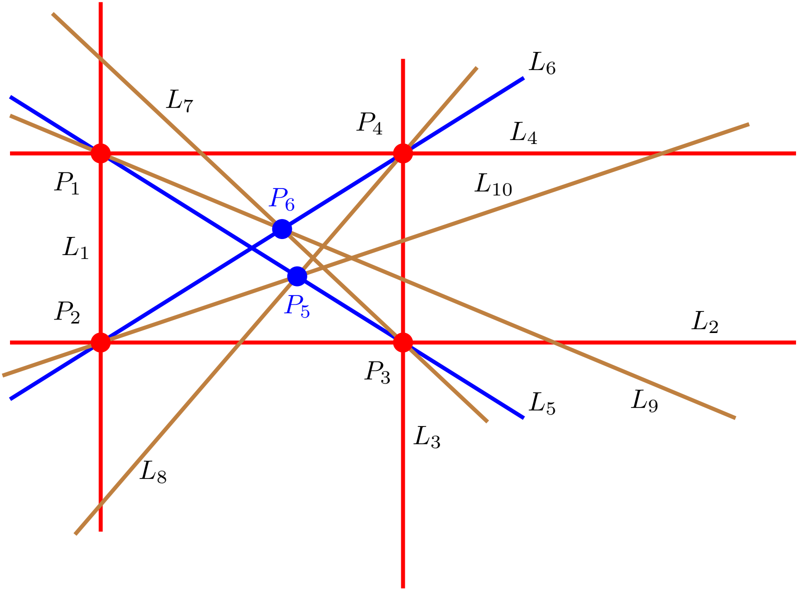

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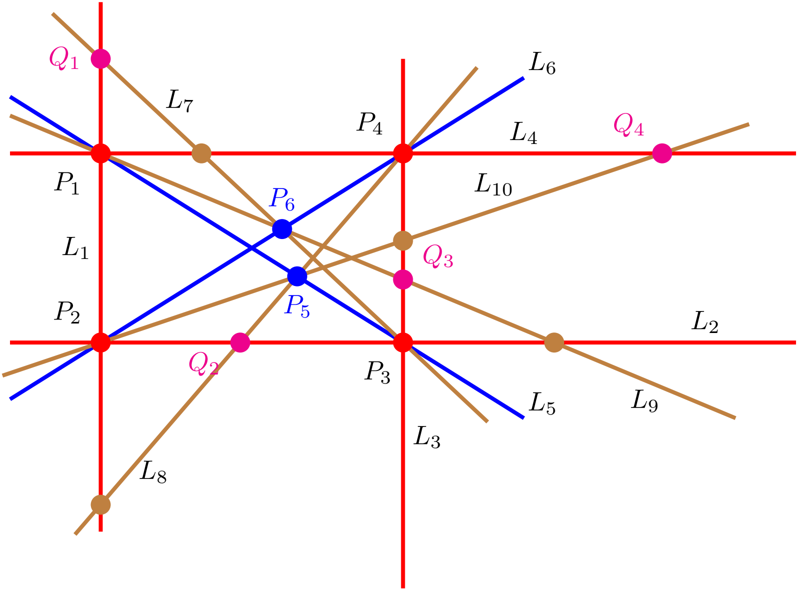

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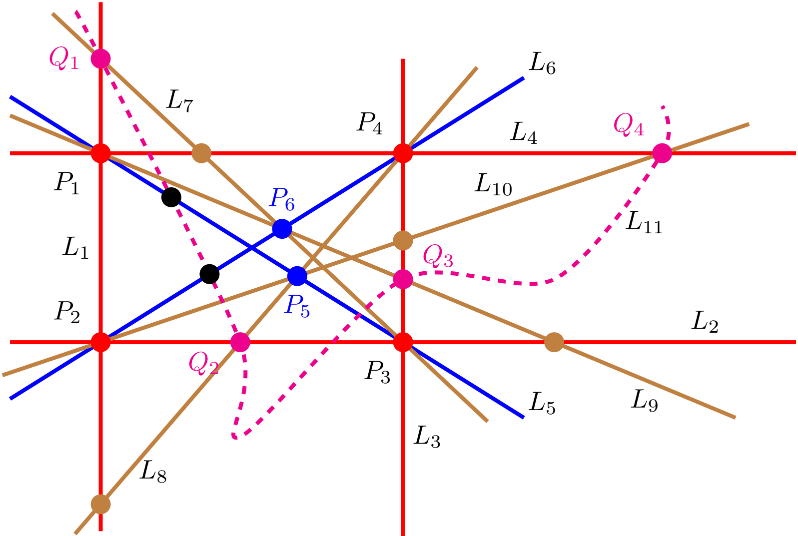

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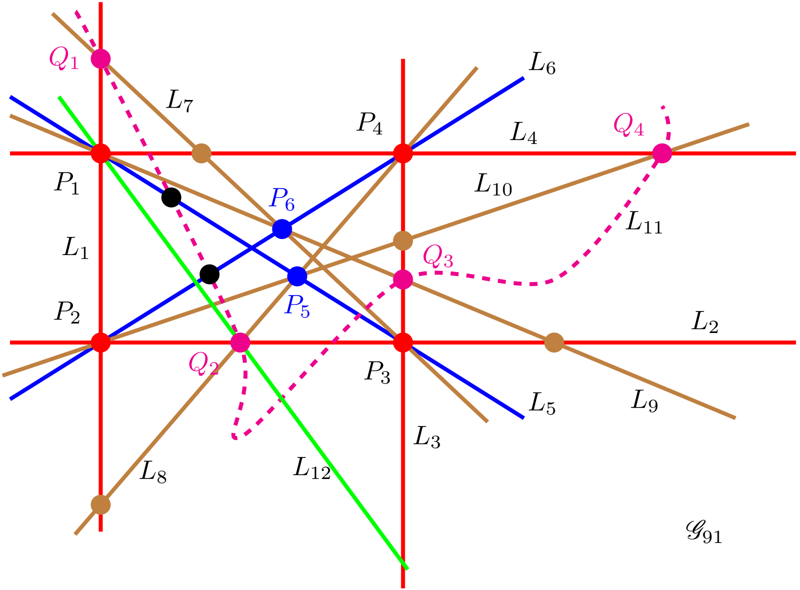

Guerville's example

Theorem

$\mathscr{M}(\mathscr{G}_{91}) = \{\mathcal{G}_{91,\zeta} \mid \zeta^4+\zeta^3+\zeta^2+\zeta+1=0\}$; they admit Galois-conjugate equations in the cyclotomic field $\mathbb{K}_5$.

There is no oriented homeomorphism $(\mathbb{P}^2, \mathcal{G}_{91,\zeta_1}) \to (\mathbb{P}^2, \mathcal{G}_{91,\zeta_2})$ if $\zeta_1 \neq \zeta_2$.

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Fundamental groups

The groups $\pi_1(\mathbb{P}^2 \setminus \mathcal{G}_{91,\zeta})$ and $\pi_1(\mathbb{P}^2 \setminus \mathcal{G}_{91,\zeta^2})$ have isomorphic profinite completions while the groups $\pi_1(\mathbb{P}^2 \setminus \mathcal{G}_{91,\zeta})$ and $\pi_1(\mathbb{P}^2 \setminus \mathcal{G}_{91,\zeta^4})$ are isomorphic.

Main result I

Theorem

The groups $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_\zeta)$ and $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{\zeta^2})$ are not isomorphic (while their profinite completions are).

First step

$\varphi : \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_\zeta) \to \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{\zeta^2})$ isomorphism $\Longrightarrow \varphi_* = \pm 1_{H_1^{\mathscr{G}_{91}}}$.

- Purely combinatorial statement.

Main result II

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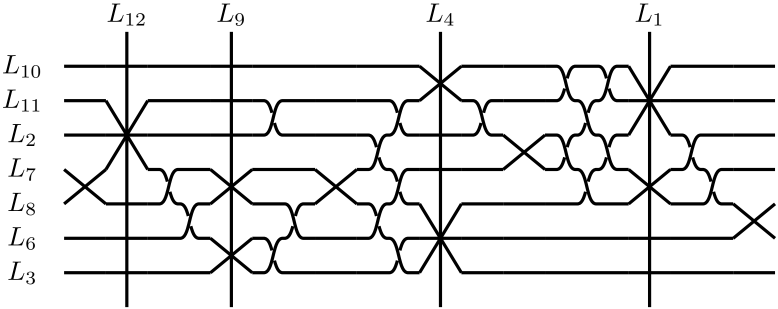
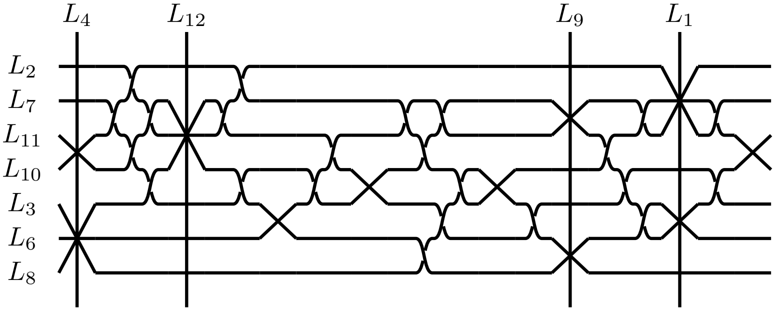
Second step

There is no isomorphism such that $\varphi : \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_\zeta) \to \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{\zeta^2})$ isomorphism $\Longrightarrow \varphi_* = 1_{H_1^{\mathscr{G}_{91}}}$

- It depends on the actual presentations of the groups.

Wiring diagrams





Some data about the proof

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- Whole process 314.02s CPU time (mostly for the construction of the system!).

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Third step

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Tack!