Pencils of curves and special curves of torus type

Enrique Artal (Universidad de Zaragoza)

Tokyo, September 2003

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Some parts with:
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Useful for obtaining properties of fundamental groups without computation

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Rational curve:

$$[t:s] \mapsto [t^2s^2 : (t+s)^2s^2 : t^2(t+s)^2].$$

L_t tangent line at P_t from $[t:1]$: $C+2L_t$ is of torus type:

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$$\begin{aligned} f(x,y,z) + 4x^3 l =& x^2y^2 + y^2z^2 + x^2z^2 - 2xyz(x+y+z) \\ &+ 4x^3(x+y+z) \\ =& \end{aligned}$$

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Double conic tangent to two cusps and passing through the contact points of the tacnode

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$$f_6(x, y, z) = 4(xy)^3 + (x^3 + y^3 - z^3)^2.$$

Theorem 3. *D special toric curve. The corresponding morphisms*

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Question 4. *What happens with the Alexander polynomial of a reduced special toric curve?*

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$$\boxed{\rho_1 : \pi_1(\mathbb{P}^2 \setminus C_6) \twoheadrightarrow \langle a, b \mid a^2 = b^3 = 1\rangle} \tag{2}$$

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$$\boxed{\rho_2 : \pi_1(\mathbb{P}^2 \setminus C_6) \twoheadrightarrow \langle a, b \mid a^2 = b^3 = (ab)^6 = 1\rangle} \tag{3}$$

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$$\boxed{\rho_i : \pi_1(\mathbb{P}^2 \setminus C_6) \twoheadrightarrow \langle a, b \mid a^2 = b^3 = 1\rangle, \quad i = 3, 4, 5.} \tag{4}$$

C is special

Theorem 5 (Oka-Pho). *The fundamental group of the complement of any irreducible tame curve of torus type is* $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$.

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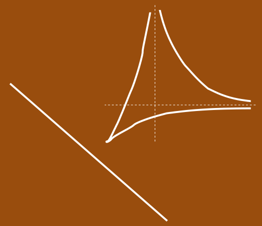
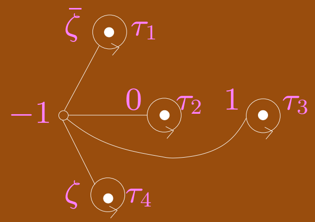
Question 6. *Under which conditions the epimorphism associated to a rational function and a curve is an isomorphism?*

5. Computation of fundamental groups



(a) Real affine picture



(b) Base for braid monodromy

Almost direct computation from the real curve (Zariski-Van Kampen with asymptotes).

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(c) *Meridians near one tacnode:* z *and* $(yzxz)^{-1}$*.*

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(c) *Meridians near one tacnode:* z *and* $(yzxz)^{-1}$.

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$$\pi_1(\mathbb{P}^2 \setminus (C \cup T)) = \langle x, y, z \mid xyx = yxy, xzx = zxz, yzy = zyz \rangle$$

Also obtained by Oka

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Get canonical epimorphism onto $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$.