Coverings of rational ruled normal surfaces

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Weighted blow-ups

Ruled surfaces

Cohomology of sheaves asssociated to divisors

Cohomology of rational ruled toric surfaces

Cyclic covers

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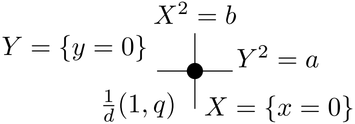

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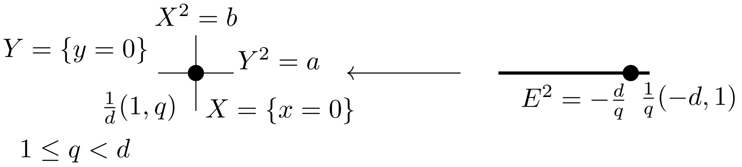

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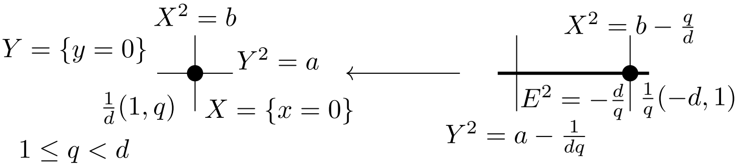

Smooth ruled rational surfaces

Hirzebruch surface Σ_n

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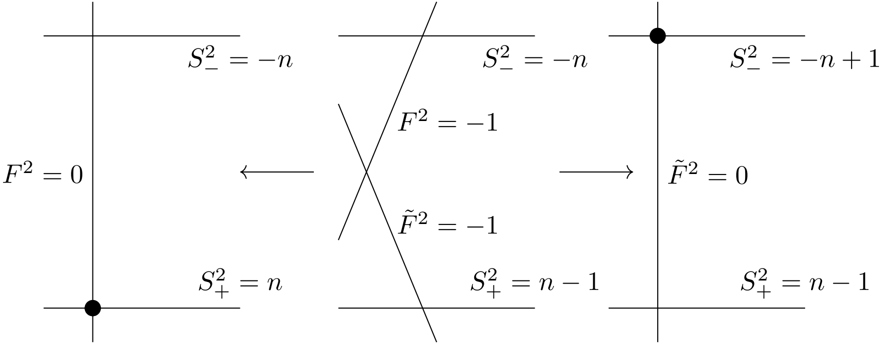

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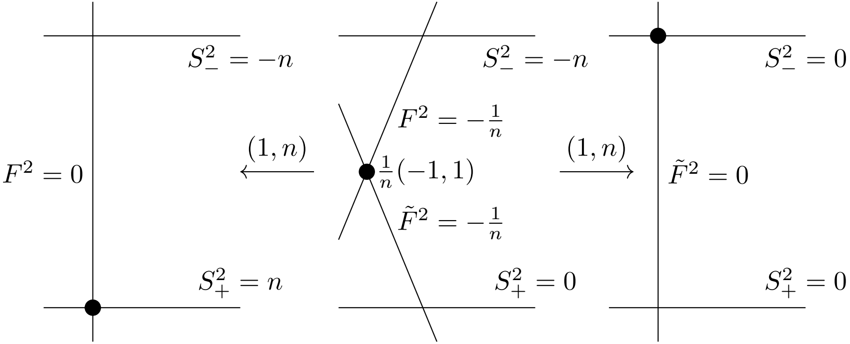

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if and only if $\frac{n_1}{d_1} + \frac{n_2}{d_2} - r \in \mathbb{Z}$. *In such a case* $S_+^2 = r$.

Proof.

Weighted Nagata transformations. □

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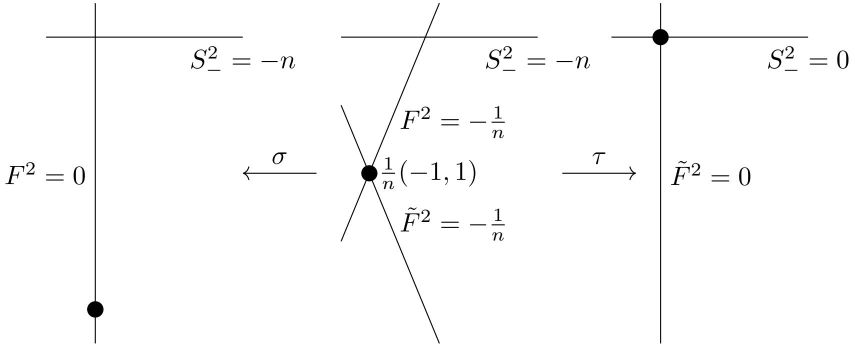
Cohomology of Σ_n

From Σ_n to Σ_0



$$D = aS_- + bF \qquad \sigma^* D = aS_- + b\tilde{F} + bF \qquad \tilde{D} = \tau_* \sigma^* D = aS_- + b\tilde{F}$$

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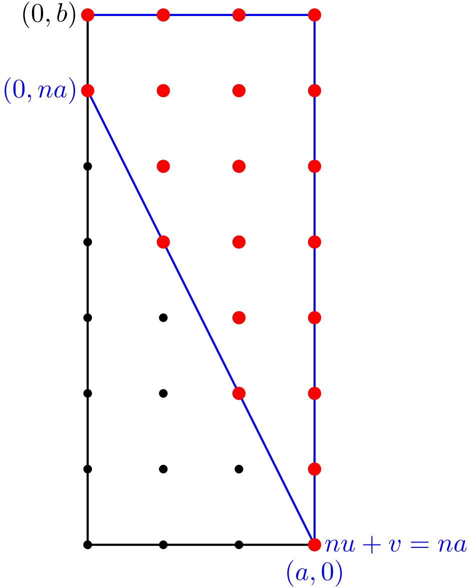

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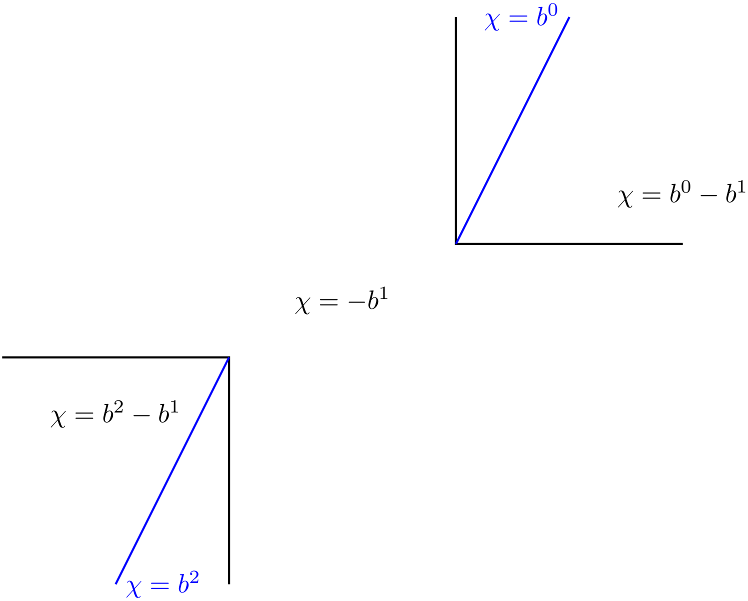
Geography of the cohomology of Σ_n



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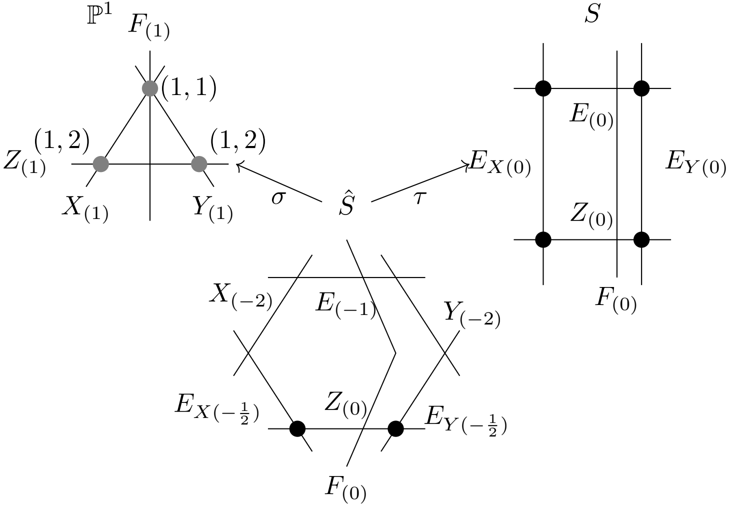
An example

Construction



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- $\mathrm{Pic}(S) = \mathbb{Z}Z \oplus \mathbb{Z}E_X \oplus \mathbb{Z}/2\mathbb{Z}(E_X - E_Y)$
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- $\tilde{D} = \tau_*(\sigma^* D) = aZ + bF$
- $\tau^* \tilde{D} - \sigma^* D = (a - \alpha)E_X + (a - \beta)E_Y + bE$
- $H^0(S; \mathcal{O}(D))$ set of $h(x, y, z) \in \mathbb{C}[x, y, z]$ such that
  - h homogeneous of degree $a + b$
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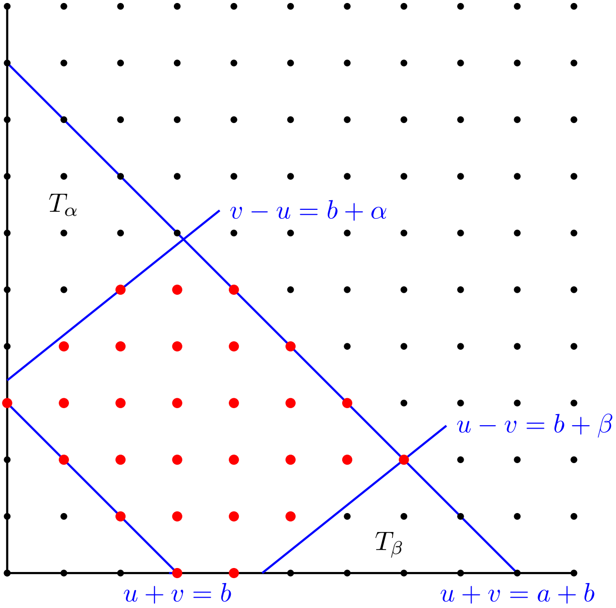
Counting points



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- When the orders of the singular points are not coprime, Pic will have torsion.

Esnault-Viehweg theory

Existence of cyclic covers

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Theorem (Esnault-Viehweg)

$$H^1(\tilde{S}; \mathcal{O}_{\tilde{S}}) = \bigoplus_{k=0}^{n-1} H^1(S; \mathcal{O}_S(L^{(k)})) \text{ where } L^{(k)} = -kB + \sum_{j=1} \left\lfloor \frac{km_i}{n} \right\rfloor D_i.$$

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- Esnault-Viehweg formula holds.

Thanks for your attention!!