Families of weighted-Yomdin singularities of surface

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Superisolated singularities

Definition

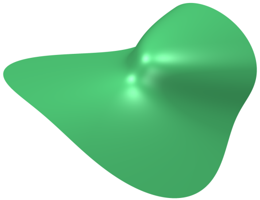
$(V,0) \subset (\mathbb{C}^3,0)$ is a superisolated singularity of surface it is solved in one blow-up

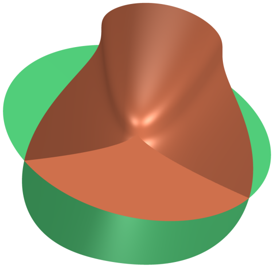
Isolated singularities $\longleftrightarrow$ projective plane curves

- $V = \{F = f_d + f_{d+1} + \cdots = 0\}$, $\boxed{\operatorname{Sing}_{\mathbb{P}}(\{f_d = 0\}) \cap \{f_{d+1} = 0\}_{\mathbb{P}} = \varnothing}$
- Luengo (1987): the μ-constant stratum is not smooth
- A_ (1991): link and characteristic polynomial of the monodromy do not determine embedded topology (Yau conjecture)
- A_, Cassou-Noguès, Luengo, Melle (2002): monodromy conjecture holds
- Fernández de Bobadilla, Luengo, Melle, Némethi (2006,2007): failure of Seiberg-Witten invariant conjecture

Superisolated singularities

Blow-up





Characteristic polynomial Stevens (1989), A__ (1991)

$$\Delta_V(t) = \frac{(t^d - 1)^{\chi(\mathbb{P}^2 \setminus C)}}{t-1} \prod_{P \in \mathrm{Sing}(C)} \Delta_{C,P}(t^{d+1})$$

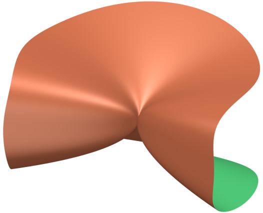
Yomdin singularities

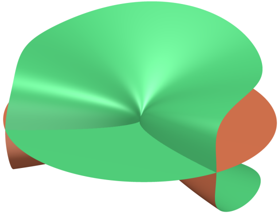
Definition

$V = \{F = f_d + f_{d+k} + \cdots = 0\}$, $\boxed{\mathrm{Sing}_{\mathbb{P}}(\{f_d = 0\}) \cap \{f_{d+k} = 0\}_{\mathbb{P}} = \varnothing}$

Gusein-Zade, Luengo, Melle (1997)

$$\Delta_V(t) = \frac{(t^d - 1)^{\chi(\mathbb{P}^2 \setminus C)}}{t-1} \prod_{P \in \mathrm{Sing}(C)} \Delta_{C,P,k}(t^{d+k}),\ \Delta_{C,P}(t) = \prod_{j=1}^{r} (1 - t^{m_j})^{n_j} \Longrightarrow \Delta_{C,P,k}(t) = \prod_{j=1}^{r} \left(1 - t^{\frac{m_j}{\gcd(k,m_j)}}\right)^{n_j \gcd(k,m_j)}$$





$\mathbf{Q}$-resolutions

Definition

Z variety with abelian quotient singularities, $D \subset Z$ is a simple $\mathbb{Q}$-normal crossing divisor if it is locally a monomial (in local coordinates where the abelian group acts diagonally).

Definition

$(V, 0) \subset (\mathbb{C}^3, 0)$ Isolated hypersurface singularity. An embedded **Q**-resolution is a proper birational map $\pi : (Z, E) \to (\mathbb{C}^3, 0)$ (isomorphism outside $\pi^{-1}(0)$) such that Z may have abelian quotient singularities and $\pi^*(V)$ is a simple $\mathbb{Q}$-normal crossing divisor.

Weighted projective planes as orbifolds

$$\omega = (p, q, r), \quad \gcd \omega = 1 \qquad \mathbb{P}^2_\omega = (\mathbb{C}^3 \setminus \{0\})/(x, y, z) \sim (t^p x, t^q y, t^r z)$$

$$\begin{aligned} \mathbb{C}^2 &\longrightarrow \{z \neq 0\} \subset \mathbb{P}^2_\omega \\ [(x, y)] &\longmapsto [x : y : 1]_\omega \end{aligned}$$

$$\omega = (p_1 d, q_1 d, r), \eta = (p_1, q_1, r) \qquad \begin{aligned} \mathbb{P}^2_\omega &\longrightarrow \mathbb{P}^2_\eta \\ [x : y : z]_\omega &\longmapsto [x : y : z^d]_\eta \end{aligned}$$

Weighted blowing-ups

$$\hat{\mathbb{C}}^3_\omega = \{(\mathbf{p}, \mathbf{u}) \in \mathbb{C}^3 \times \mathbb{P}^3_\omega \mid \mathbf{p} \in \mathbf{u}\}$$

$$\begin{aligned} \mathbb{C}^3 &\longrightarrow \{z \neq 0\} \subset \hat{\mathbb{C}}^3_\omega \\ [(x, y, z)] &\longmapsto ((xz^p, yz^q, z^r), [x : y : 1]_\omega) \end{aligned}$$

Weighted-Yomdin singularities. Definitions

Definition

- ω weight, $\gcd \omega = 1$
- $F = f_d + f_{d+k} + \ldots$, $f_d. \neq 0$, decomposition in ω-quasi-homogeneous forms
- $V := F^{-1}(0)$ is weighted-Yomdin singularity if $\{f_d\}_{\mathbb{P}_\omega}$ and $\{f_{d+k}\}_{\mathbb{P}_\omega}$ intersect *generically*: if $f_d(x_0, y_0.z_0) = f_{d+k}(x_0, y_0, z_0) = 0$, then

$$\operatorname{ord}_{(x_0,y_0,z_0)} f_d = 1.$$

- Milnor number can be expressed in terms of ω, d and of Milnor numbers of $\operatorname{Sing}\{f_d = 0\}_{\mathbb{P}_\omega}$

Weighted-Yomdin singularities. Applications

- $f_{12} = Z^{12} + XY^3Z + tX^3Y^2 + X^6$ ω-qh of degree 12, $\omega = (2,3,1)$.
- $\mathrm{Sing}\{f_{12} = 0\} = \{[0:1:0]_\omega\} \subset \mathbb{P}^2_\omega$.
- In $\frac{1}{3}(2,1)$: $f_t = z^{12} + xz + tx^3 + x^6$, two smooth extremal branches.
- Topologically equisingular
- Intersection number $X = 0$ with the Z-branch depend on t.
- $F = f_{12} + Y^5$ weighted-Yomdin, $k = 3$.
- Tangent cone changes

Weighted-Yomdin singularities. Prehistory

Briançon-Speder (1975) (isolated quasi-homogeneous $\Longrightarrow$ weighted-Yomdin)

- $\omega = (1,2,3)$, $F = Z^5 + tY^6Z + XY^7 + X^{15}$, smooth curve in $\mathbb{P}^2_\omega$, $P := [0:1:0]_\omega$, $(\mathbb{P}^2_\omega, P) = \frac{1}{2}(1,1)$, for $t = 0$ the tangent line is $X = 0$.

Remark

Since the families are μ-constant and of the type $f + tg$, topological triviality comes from Parusiński (1999).

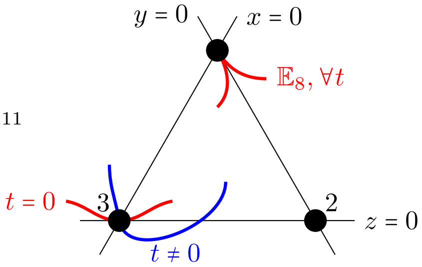
Weighted-Yomdin singularities. Non-linear families, $\omega = (2,3,1)$

- $f_{12} = (y^2 + x^3)^2 + (yz + tx^2)^3$
- $\mathrm{Sing}\{f_{12} = 0\}_{\mathbb{P}_\omega} = \{P = [0:0:1]_\omega, Q = [-1:1:-t]_\omega\}$
- $(\mathbb{P}^2_\omega, P)$ smooth, $(\{f_{12} = 0\}_{\mathbb{P}_\omega}, P)$ topologically as $u^3 = v^6$.
- $(\mathbb{P}^2_\omega, Q)$ smooth, $(\{f_{12} = 0\}_{\mathbb{P}_\omega}, Q)$ topologically as $u^3 = v^2$.
- $f_{13} = Z^{13} + X^2 Y^3$. If $t^{13} + 1 \neq 0$, $f_{12} + f_{13} = 0$ is weighted-Yomdin, $k = 1$.
- Characteristic polynomial can be computed mixing Gusein-Zade, Luengo, Melle (1997) and Martín-Morales (2013):

Rational Homology Spheres

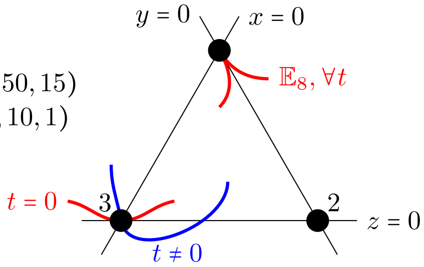
$$F = y^3 z + t x^2 y^2 + x^5 + z^{11}$$

$$\omega = (2, 3, 1)$$



$$\omega = (33, 50, 15)$$

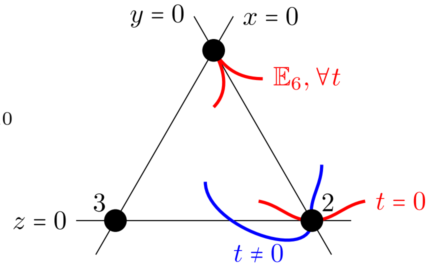
$$\eta = (11, 10, 1)$$



More rational homology spheres

$$F = y^3 + x^4 z + t x^3 y + z^{10}$$

$$\omega = (2, 3, 1)$$



Further problems