Improper integrals, Bernstein-Sato polynomial and Yano's conjecture

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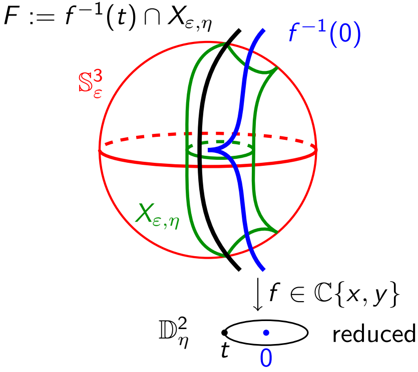
Encuentro de la Red de
Geometría Algebraica y Singularidades 2018
Universidad Complutense
February 1st 2018

Joint work with I. Luengo, P. Cassou-Noguès and A. Melle

Plane curve singularities



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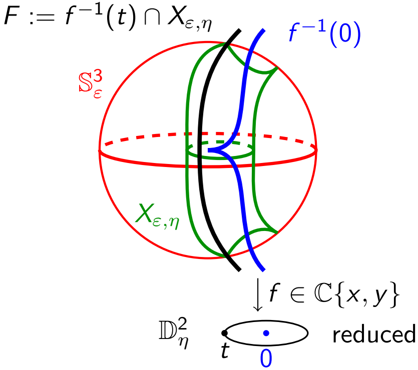

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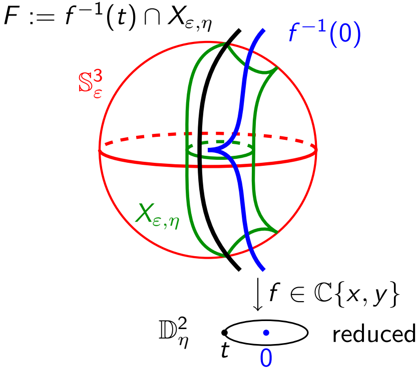

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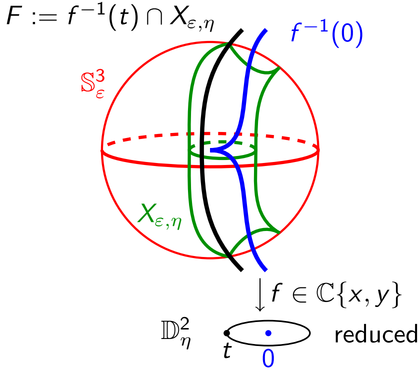

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Theorem (Monodromy Theorem)

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- Via $\mathcal{D}$-module theory (K) or Gauss-Manin theory (M), construct a $\mathbb{C}$-vector space H of dimension μ and an endomorphism $\varphi : H \to H$ such that $\exp(-2i\pi\varphi)$ is conjugated to ρ^*

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Differential operators

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- $\mathcal{D}[s]$ is the ring of differential operators

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- Reduced Bernstein-Sato polynomial $\tilde{b}_f(s) := \frac{b_f(s)}{s+1}$; $\tilde{b}_f(-s)$ is the minimal polynomial of φ.

Some features of Bernstein-Sato polynomial

- *Computable:* Gauss-Manin (M. Schulze), non-commutative Groebner basis (Levandovskyy, Martín-Morales) in `Singular`, `Asir`.
  Hard to compute.

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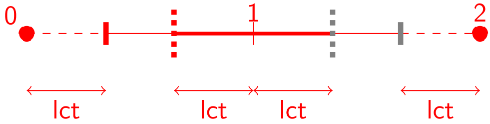

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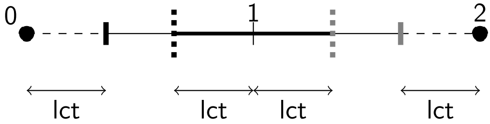

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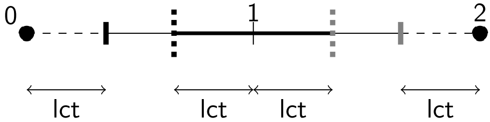

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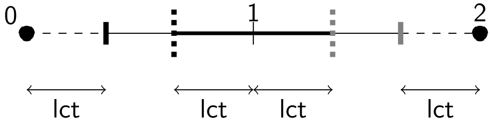

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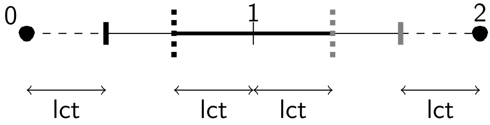

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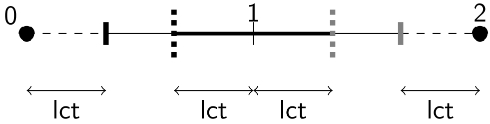

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- Varchenko's upper semicontinuity if monodromy has not multiple eigenvalues ($\neq 1$)

Improper integrals: one variable

Hypothesis

$f : [0,1] \to \mathbb{R}$ function $\mathcal{C}^\infty$, $f > 0$ in $[0,1]$, $a, b \in \mathbb{N}$

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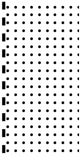
Holomorphic if $\Re s > -\frac{b}{a}$



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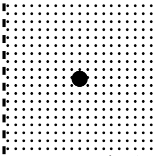
Meromorphic continuacion to
$\Re s > -\frac{b+1}{a}$ with a pole in $-\frac{b}{a}$



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Conclusion

$\mathcal{I}(s)$ Meromorphic with poles in $\mathbb{Q}_{<0}$ of order 1

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f $\overline{\mathbb{Q}}_{\mathbb{R}}$-analytic for $x^{\frac{1}{N}}$ in $[0,1]$, $f > 0$ en $(0,1]$, $a, b \in \mathbb{N}$, $g \in \overline{\mathbb{Q}}_{\mathbb{R}}[x,s]$ defined in $[0,1] \times \mathbb{C}$

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Question

What happens if $f > 0$ in $[0,1]^2 \setminus \{(0,0)\}$? Study f *near* the origen.

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Neighborhood of $f = y^5 + x^4y^2 + x^7$ in its μ-constant stratum

$f_t(x, y) := f + tx^6y$, $t \in \overline{\mathbb{Q}}_{\mathbb{R}}$, ☡

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$$\mathcal{I}(s) := \int_0^1\int_0^1 f_t(x, y)^s x^{\beta_1} y^{\beta_2} \frac{dx}{x}\frac{dy}{y} = \overset{\mathcal{I}_1}{\int} + \overset{\mathcal{I}_2}{\int} \qquad y = x^{\frac{4}{3}}$$

Example I

Neighborhood of $f = y^5 + x^4y^2 + x^7$ in its μ-constant stratum

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$$\mathcal{I}_2(s) \underset{x\mapsto xy^3}{\overset{y\mapsto y^4}{=}} 4\int_0^1\int_0^1 \left(1 + x^4 + x^7y + tx^6y^2\right)^s x^{\beta_1} y^{20s+3\beta_1+4\beta_2} \frac{dx}{x}\frac{dy}{y}$$

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$$\mathcal{I}_1(s) \overset{y\mapsto x^4y}{\underset{x\mapsto x^3}{=}} 3\int \left(y^5 + y^2 + x + tx^2y\right)^s x^{20s+3\beta_1+4\beta_2} y^{\beta_2} \frac{dx}{x}\frac{dy}{y} = \mathcal{I}_{11}(s) + \mathcal{I}_{12}(s)$$

$y = x^{\frac{1}{2}}$

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Neighborhood of $f = y^5 + x^4y^2 + x^7$ in its μ-constant stratum

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$$\mathcal{I}_1(s) \underset{x\mapsto x^3}{\overset{y\mapsto x^4y}{=}} 3\int_{\square}\left(y^5+y^2+x+tx^2y\right)^s x^{20s+3\beta_1+4\beta_2} y^{\beta_2}\frac{dx}{x}\frac{dy}{y} = \mathcal{I}_{11}(s)+\mathcal{I}_{12}(s)$$

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$$\mathcal{I}_{11}(s) \underset{x\mapsto x^2}{\overset{y\mapsto xy}{=}} 6\int_{\square}\left(x^3y^5+y^2+1+tx^3y\right)^s x^{3(14s+2\beta_1+3\beta_2)} y^{\beta_2}\frac{dx}{x}\frac{dy}{y}$$

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$$\mathcal{I}_{12}(s) \underset{x\mapsto x^2y}{=} 3\int_{\square}\left(y^3+1+x+tx^2y^3\right)^s x^{20s+3\beta_1+4\beta_2} y^{3(14s+2\beta_1+3\beta_2)}\frac{dx}{x}\frac{dy}{y}$$

Example II

$$f_t(x,y) = y^5 + x^4y^2 + x^7 + tx^6y$$

$$\mathcal{I}(s) = 4\int_0^1\int_0^1 \left(1 + x^4 + x^7y + tx^6y^2\right)^s x^{\beta_1} y^{20s+3\beta_1+4\beta_2} \frac{dx}{x}\frac{dy}{y} +$$

$$6\int_{\square}\left(x^3y^5+y^2+1+tx^3y\right)^s x^{3(14s+2\beta_1+3\beta_2)} y^{\beta_2}\frac{dx}{x}\frac{dy}{y} +$$

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Residues for $\alpha = -\frac{3\beta_1+4\beta_2}{20}$

$$\operatorname*{Res}_{s=\alpha} = 4\int_0^1 (1+x^4)^{-\alpha} x^{\beta_1}\frac{dx}{x} + 3\int_0^1 (1+y^3)^{-\alpha} y^{3\frac{2\beta_2-\beta_1}{10}}\frac{dy}{y}$$

Example II

$$f_t(x,y) = y^5 + x^4y^2 + x^7 + tx^6y$$

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Residues for $\alpha = -\frac{3\beta_1+4\beta_2}{20}$

$$\operatorname*{Res}_{s=\alpha} = \int_0^1 (1+u)^{-\alpha} u^{\frac{\beta_1}{4}} \frac{du}{u} + \int_0^1 (1+u)^{-\alpha} u^{\frac{2\beta_2-\beta_1}{10}} \frac{du}{u} = \boldsymbol{B}\left(\frac{\beta_1}{4}, \frac{2\beta_2-\beta_1}{10}\right)$$

Example II

$$f_t(x,y) = y^5 + x^4y^2 + x^7 + tx^6y$$

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Residues for $\alpha = -\frac{9}{20}$, $\beta_i = 1$.

Involves $\frac{\partial^2 f_t^{-\frac{9}{20}}}{\partial x^2}(0,y)$ and $\frac{\partial^2 f_t^{-\frac{9}{20}}}{\partial y^2}(x,0)$

$$\operatorname*{Res}_{s=-\frac{9}{20}} = \frac{9\cdot 29}{4000}\int_0^1 (1+x^4)^{-\frac{49}{20}}x^{15}\frac{dx}{x} - \frac{9t}{100}\int_0^1(1+x^4)^{-\frac{29}{20}}x^7\frac{dx}{x}$$

$$+\frac{3\cdot 9\cdot 29}{16000}\int_0^1 (1+y^3)^{-\frac{49}{20}}y^{-\frac{39}{10}}\frac{dy}{y} - \frac{27t}{400}\int_0^1 (1+y^3)^{-\frac{29}{20}}y^{-\frac{9}{10}}\frac{dy}{y}$$

Example II

$$f_t(x,y) = y^5 + x^4y^2 + x^7 + tx^6y$$

$$\mathcal{I}(s) = 4\int_0^1\int_0^1 \left(1 + x^4 + x^7y + tx^6y^2\right)^s x^{\beta_1} y^{20s+3\beta_1+4\beta_2}\frac{dx}{x}\frac{dy}{y} +$$

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$$\operatorname*{Res}_{s=-\frac{9}{20}} = \frac{9\cdot 29}{16000}\boldsymbol{B}\left(\frac{15}{4}, -\frac{13}{10}\right) - \frac{9t}{400}\boldsymbol{B}\left(\frac{7}{4}, -\frac{3}{10}\right) = \frac{208t+385}{1664}\boldsymbol{B}\left(\frac{3}{4}, \frac{7}{10}\right)$$

Strategy

$$f^s = \frac{1}{b_f(s)} P \cdot f^{s+1}$$

Strategy

$$f^s x^{\beta_1-1} y^{\beta_2-1} = \frac{1}{b_f(s)} P \cdot f^{s+1} x^{\beta_1-1} y^{\beta_2-1}$$

Strategy

$$\mathcal{J}_{\beta_1,\beta_2}(s) := \int_0^1\int_0^1 f^s x^{\beta_1-1} y^{\beta_2-1} dxdy = \frac{1}{b_f(s)} \int_0^1\int_0^1 P \cdot f^{s+1} x^{\beta_1-1} y^{\beta_2-1} dxdy$$

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Theorem

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Theorem

- $\alpha \in \mathbb{Q}_{>0}$

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Theorem

- $\alpha \in \mathbb{Q}_{>0}$
- $-\alpha$ *pole of* $\mathcal{J}_{\beta_1,\beta_2}(s)$ *with transcendental residue.*

Strategy

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- $\mathcal{J}_{\beta_1',\beta_2'}(s)$, $\beta_i' \geq 1$, *is holomorphic at* $-(\alpha - 1)$

Strategy

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- *Then,* $-\alpha$ *is a root of the Bernstein-Sato polynomial.*

Strategy

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Theorem

- $\alpha \in \mathbb{Q}_{>0}$
- *$-\alpha$ pole of $\mathcal{J}_{\beta_1,\beta_2}(s)$ with transcendental residue.*
- *$\mathcal{J}_{\beta_1',\beta_2'}(s)$, $\beta_i' \geq 1$, is holomorphic at $-(\alpha-1)$*
- *Then, $-\alpha$ is a root of the Bernstein-Sato polynomial.*

Rough sketch of the proof.

Integration by parts $\Rightarrow \int_{\square} \overbrace{x^{\beta_1'-1}}^{V} y^{\beta_2'-1} \overbrace{\frac{\partial^{u+v} f^s}{\partial x^u \partial y^v}_{|(x,y)} dxdy}^{dU} = (\beta_1'-1)$

$$\left(\int_0^1 y^{\beta_2'-1} \frac{\partial^{u+v-1} f^s}{\partial x^{u-1}\partial y^v}_{|(1,y)} dy - \int_{\square} x^{\beta_1'-2} y^{\beta_2'-1} \frac{\partial^{u+v-1} f^s}{\partial x^{u-1}\partial y^v}_{|(x,y)} dxdy\right)$$

Strategy

$$\mathcal{J}_{\beta_1,\beta_2}(s) := \int_0^1\int_0^1 f^s x^{\beta_1-1} y^{\beta_2-1} dxdy = \frac{1}{b_f(s)} \int_0^1\int_0^1 P\cdot f^{s+1} x^{\beta_1-1} y^{\beta_2-1} dxdy$$

Theorem

- *$\alpha \in \mathbb{Q}_{>0}$*
- *$-\alpha$ pole of $\mathcal{J}_{\beta_1,\beta_2}(s)$ with transcendental residue.*
- *$\mathcal{J}_{\beta_1',\beta_2'}(s)$, $\beta_i' \geq 1$, is holomorphic at $-(\alpha-1)$*
- *Then, $-\alpha$ is a root of the Bernstein-Sato polynomial.*

Rough sketch of the proof.

$$2^{\text{nd}} \text{ factor of RHS} \begin{cases} \text{holomorphic or} \\ \text{simple pole with algebraic residue} \end{cases} +$$

$$\text{LHS simple pole with transcendental residue} \Longrightarrow$$

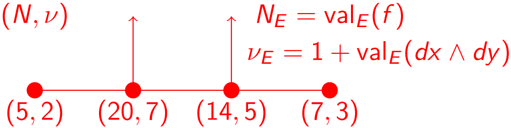
$$2^{\text{nd}} \text{ factor of RHS holomorphic} + 1^{\text{st}} \text{ factor of RHS simple pole}$$

Example III

Dual graph

$\{ 1 \}$

$\left\{ \frac{5}{14}, \frac{1}{2}, \frac{9}{14}, \frac{11}{14}, \frac{13}{14}, \frac{15}{14}, \frac{17}{14} \right\}$



$\left\{ \frac{7}{20}, \frac{9}{20}, \frac{1}{2}, \frac{11}{20}, \frac{13}{20}, \frac{7}{10}, \frac{3}{4}, \frac{17}{20}, \frac{9}{10}, \frac{19}{20}, \frac{21}{20}, \frac{11}{10}, \frac{23}{20}, \frac{5}{4}, \frac{13}{10} \right\}$

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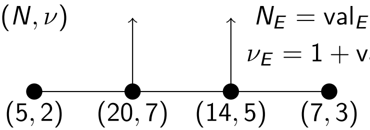
Dual graph



$N_E = \mathrm{val}_E(f)$

$\nu_E = 1 + \mathrm{val}_E(dx \wedge dy)$

$\left\{\frac{7}{20}, \frac{9}{20}, \frac{1}{2}, \frac{11}{20}, \frac{13}{20}, \frac{7}{10}, \frac{3}{4}, \frac{17}{20}, \frac{9}{10}, \frac{19}{20}, \frac{21}{20}, \frac{11}{10}, \frac{23}{20}, \frac{5}{4}, \frac{13}{10}\right\}$

- Spec $\cap [0,1] = \left\{\frac{7}{20}, \frac{11}{20}, \frac{7}{10}, \frac{3}{4}, \frac{17}{20}, \frac{9}{10}, \frac{19}{20}\right\} \cup \left\{\frac{1}{2}, 1\right\} \cup \left\{\frac{9}{14}, \frac{11}{14}, \frac{13}{14}\right\}$

Example III

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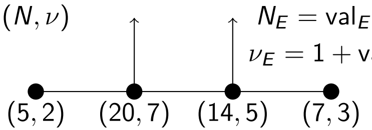
Dual graph



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$\left\{\frac{7}{20}, \frac{9}{20}, \frac{1}{2}, \frac{11}{20}, \frac{13}{20}, \frac{7}{10}, \frac{3}{4}, \frac{17}{20}, \frac{9}{10}, \frac{19}{20}, \frac{21}{20}, \frac{11}{10}, \frac{23}{20}, \frac{5}{4}, \frac{13}{10}\right\}$

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- $\frac{1}{2}$ double root (Jordan 2-block)

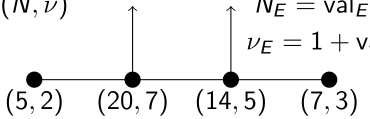
Example III

$\{1\}$ $\quad\left\{\frac{5}{14}, \frac{1}{2}, \frac{9}{14}, \frac{11}{14}, \frac{13}{14}, \frac{15}{14}, \frac{17}{14}\right\}$

Dual graph (N, ν)

$N_E = \mathrm{val}_E(f)$

$\nu_E = 1 + \mathrm{val}_E(dx \wedge dy)$



$$\left\{\frac{7}{20}, \frac{9}{20}, \frac{1}{2}, \frac{11}{20}, \frac{13}{20}, \frac{7}{10}, \frac{3}{4}, \frac{17}{20}, \frac{9}{10}, \frac{19}{20}, \frac{21}{20}, \frac{11}{10}, \frac{23}{20}, \frac{5}{4}, \frac{13}{10}\right\}$$

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- $\frac{1}{2}$ double root (Jordan 2-block)
- $\frac{13}{20}, \frac{21}{20}, \frac{15}{14}, \frac{11}{10}, \frac{23}{20}, \frac{17}{14}, \frac{5}{4}, \frac{13}{10} \in \left[\frac{13}{20}, \frac{27}{20}\right) \Longrightarrow$ roots of $\tilde{b}_{f_t}(-s)$.

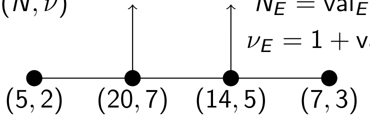
Example III

$\{1\}$ $\quad \left\{\frac{5}{14}, \frac{1}{2}, \frac{9}{14}, \frac{11}{14}, \frac{13}{14}, \frac{15}{14}, \frac{17}{14}\right\}$

Dual graph (N, ν)

$N_E = \mathrm{val}_E(f)$

$\nu_E = 1 + \mathrm{val}_E(dx \wedge dy)$



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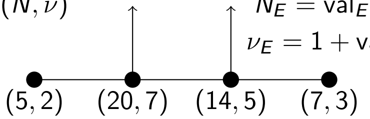

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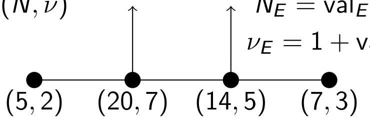

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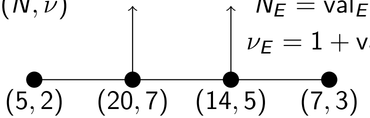

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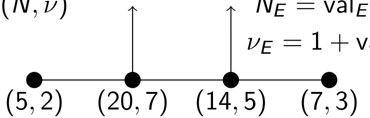

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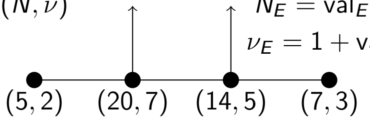

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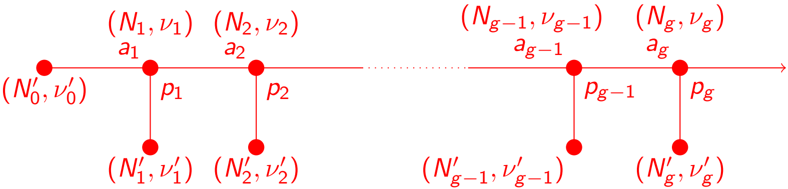
Yano's Conjecture

Dual graph of an irreducible germ



$$R(t) := t + \sum_{k=1}^{g} t^{\frac{\nu_k}{N_k}} \frac{1-t}{1-t^{\frac{1}{N_k}}} - \sum_{k=0}^{g} t^{\frac{\nu'_k}{N'_k}} \frac{1-t}{1-t^{\frac{1}{N'_k}}}$$

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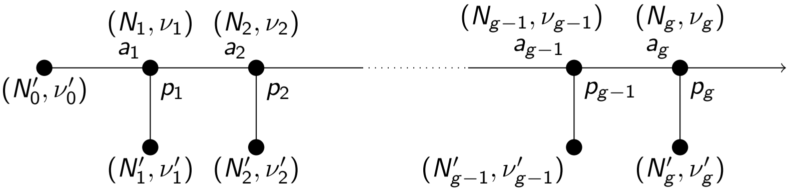

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B_f for f generic in a μ-constant stratum with resolution graph as above is formed by the exponents of the monomials of $R(t)$ (counted with multiplicity).

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Yano's conjecture holds for one-Puiseux pair singularities.

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Future

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Singular thanks for your attention!!