How can one check if a tuple of curves is a Zariski tuple?

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Irreducible Zariski tuple candidates

__, J.I. Cogolludo, and J. Martín. "Triangular Curves and Cyclotomic Zariski Tuples." **Collect. Math. 71** (3) 2020, 427–41

- $\mathscr{M}^{2d}_{3(d,d+1)}$: $\mathcal{C}_{2d} \subset \mathbb{P}^2$, $\#\operatorname{Sing}(\mathcal{C}_{2d}) = 3$, singularities $\overset{\text{top}}{\sim} \{u^d = v^{d+1}\}$.
- Are two such curves topologically equivalent in $\mathbb{P}^2$?
- $d = 2$: tricuspidal quartic, projectively equivalent.
- $d = 3$: sextic with 3 $\mathbb{E}_6$, topology depends on the existence of a conic tangent to the singularities.
  - Distinct fundamental groups
  - Distinct Alexander polynomials
- $d = 4$: a general member (e.g, $a = 1, b = -1, c = 0$) of

$$y^4(x+z)^4 + x^4(y+z)^4 + z^4(x+\lambda y)^4 - (y^4z^4 + x^4z^4 + x^4y^4)$$
$$+4x^2y^2z^3(ax+by) + cx^3y^3z^2 = 0, \qquad a, b, c \in \mathbb{C},$$
$$\lambda^4 = 1 \qquad \lambda = 1, -1, i.$$

Topology and algebra of algebraic curves

Embedded homeomorphism types

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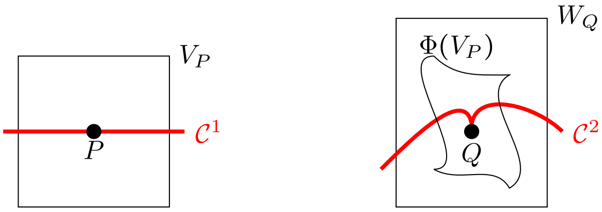

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- π_1^{loc} with peripheral invariants does determine the local topological type of $(\mathcal{C}^1, P)$.

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Definition (Combinatorics)

$\mathcal{C} \subset \mathbb{P}^2$, $\sigma : X \to \mathbb{P}^2$ minimal embedded resolution of $\operatorname{Sing}^1(\mathcal{C})$ (take out nodes with brances in distinct global irreducible components).

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- dual graph of $\sigma^{-1}(\mathcal{C})$;
- weighted with genus and self-intersections;
- strict transforms of irreducible components of $\mathcal{C}$ are marked.

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If $\mathcal{C}^1, \mathcal{C}^2 \subset \mathbb{P}^2$ share combinatorics $\exists \Phi : (\mathcal{T}(\mathcal{C}^1), \mathcal{C}^1) \to (\mathcal{T}(\mathcal{C}^2), \mathcal{C}^2)$ homeomorphism for suitable regular neighborhoods.

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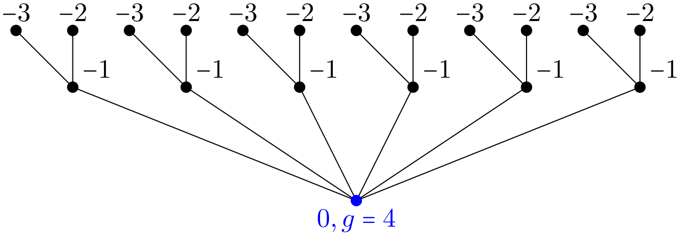

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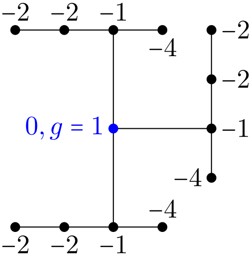

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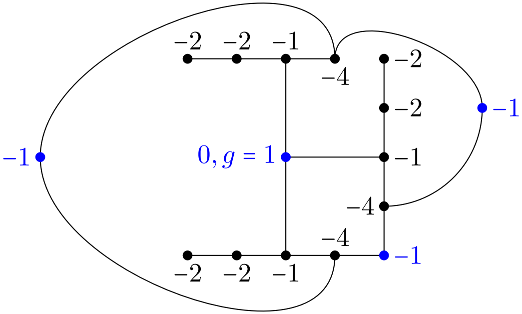

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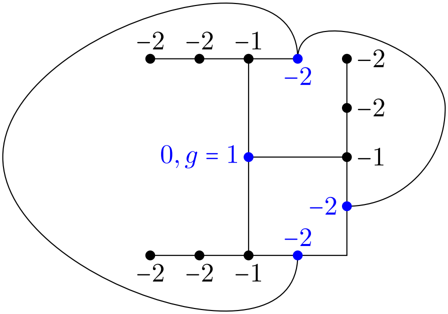

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Purely algebraic methods cannot work!

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- Zariski pairs of complexified real arrangements by Guerville and Viu (algebraic properties, conics passing throught multiple points or tangent to lines).

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Complements and fundamental groups

Oka-Eyral curves

Arithmetic Zariski pair with isomorphic complements ($\mathbb{Q}(\sqrt{3})$)

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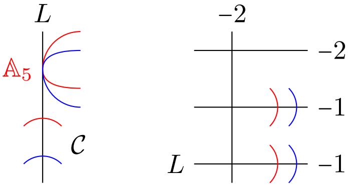

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In most cases if $\mathcal{C}_1, \mathcal{C}_2$ form a Zariski pair then $\mathbb{P}^2 \setminus \mathcal{C}_i$ are not homeomorphic.

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Theorem (_,Cogolludo, Martín)

There are Zariski tuples (distinguished by linking invariant) with abelian fundamental groups.

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$\mathscr{M}^{2d}_{3(d,d+1)}$

Set of projective plane curves in $\mathbb{P}^2 := \mathbb{CP}^2$ of degree d with three singular types of topological type as $v^d - u^{d+1} = 0$.

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- Use diagonal-permutation automorphisms.

Realization Space II

Finding equations

$\mathcal{C} = \{F_{2d}(X,Y,Z) = 0\}$, $Z = 0$ not tangent line for $(\mathcal{C}, [0:1:0]) \Longrightarrow$ coefficient of $(XY)^d$ is $\neq 0$ ($\Rightarrow$ is 1)

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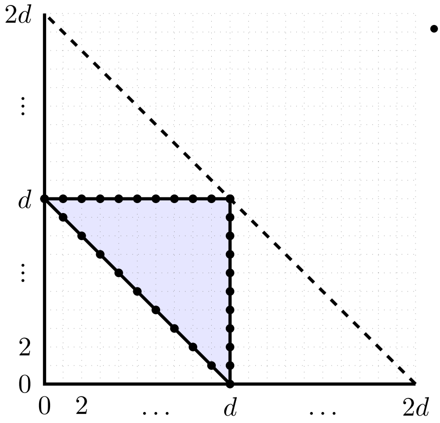

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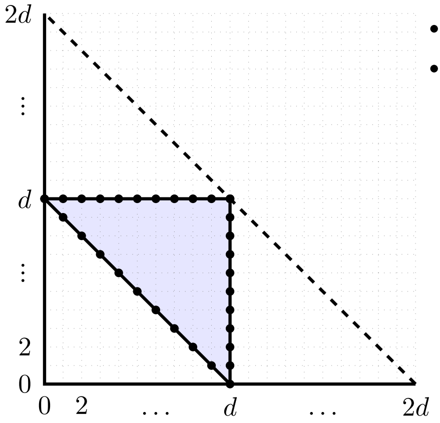

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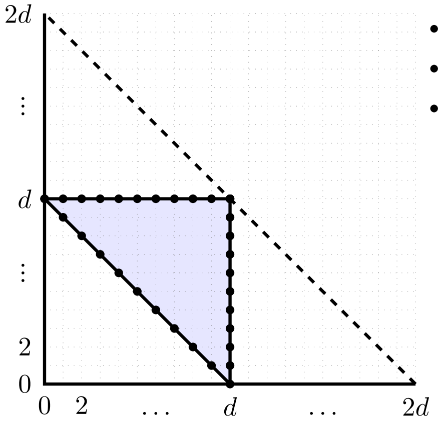

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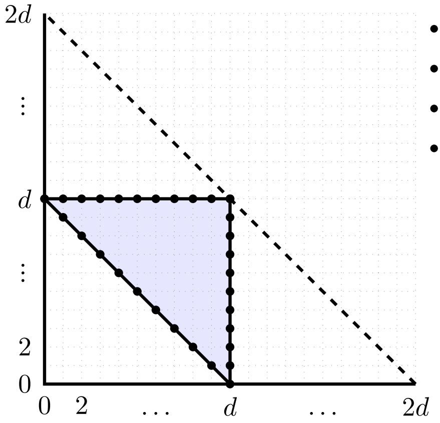

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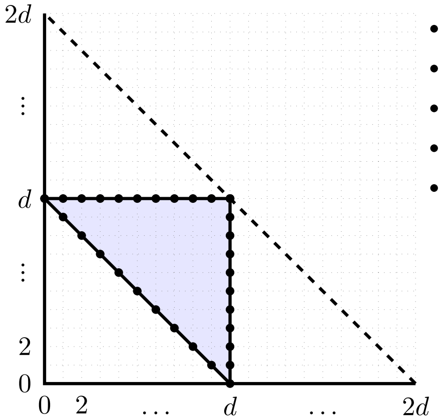

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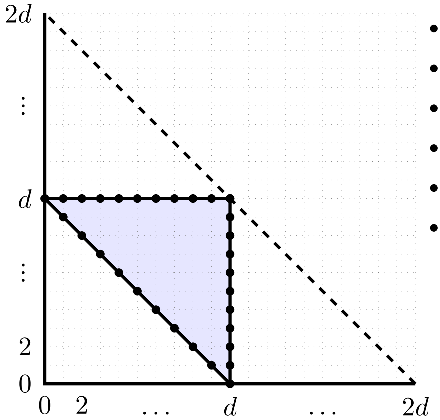

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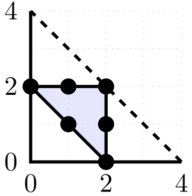
Realization Space III

Decomposition via roots of unity

$\mathscr{M}^{2d}_{3(d,d+1)}$ decomposes in $\left\lfloor \frac{d}{2} \right\rfloor + 1$ subsets $\mathscr{M}^{2d}_{3(d,d+1)}(\omega)$ parametrized by the sets $\{\omega, \omega^{-1}\}$, when $\omega^d = 1$.

Small degrees

$d = 2$



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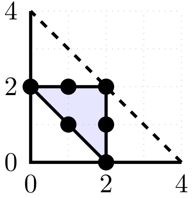

$\boxed{\omega = 1}$ $(YZ + XZ + XY)^2$, $\nexists \mathcal{C}$

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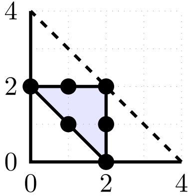

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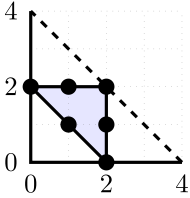

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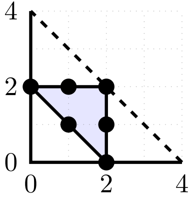

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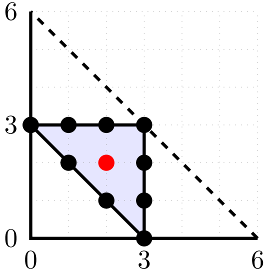

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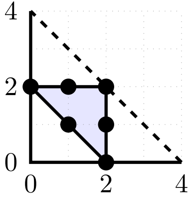

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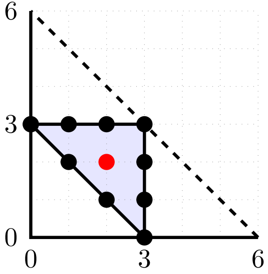

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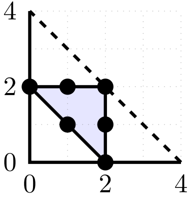

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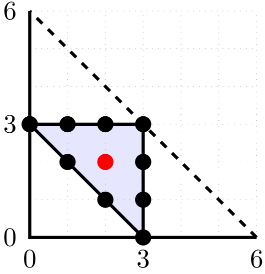

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Theorem

If $d \geq 3$, $\mathscr{M}^{2d}_{3(d,d+1)}$ *has* $\left\lfloor \frac{d}{2} \right\rfloor + 1$ *connected components parametrized by the sets* $\{\omega, \omega^{-1}\}$, *when* $\omega^d = 1$.

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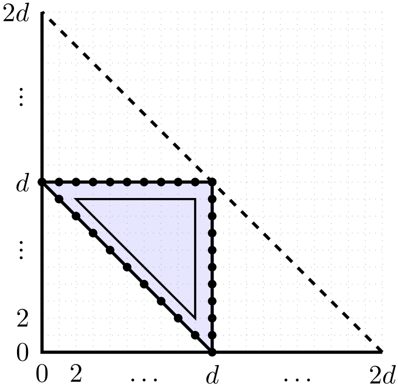
Proof.



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$XYZG_\omega(X, Y, Z) = 0$, smooth curve with three maximal flexes.

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- $G_\omega(0,Y,Z) = (Y+Z)^d, G_\omega(X,0,Z) = (X+Z)^d$
- $G_\omega(X,Y,0) = (Y+\omega X)^d$
- $H_\omega = \{[X:Y:Z:T] \in \mathbb{P}^3 \mid T^d = G_\omega(X,Y,Z)\} \xrightarrow{\rho_\omega} \mathbb{P}^2$.

Cremona transformation and algebraic computation of the linking invariant

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- $\rho_\omega^{-1}(\mathcal{L}_a)=\bigcup_{\zeta^d=1}\mathcal{L}_a^\zeta$, $\mathcal{L}_a^\zeta=\{X=0,T=\zeta(Y+Z)\}$

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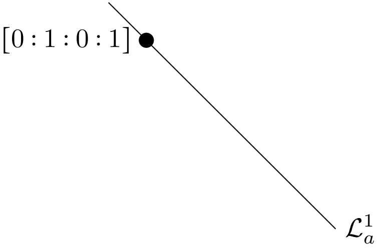

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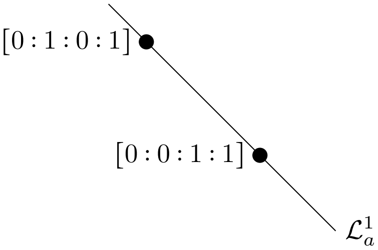

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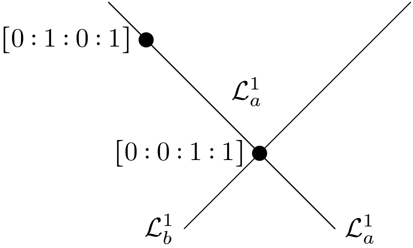

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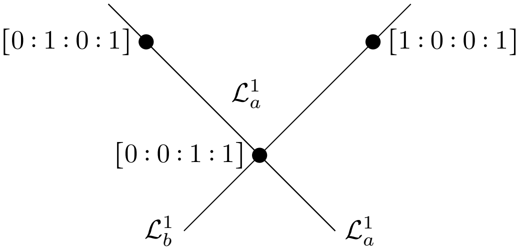

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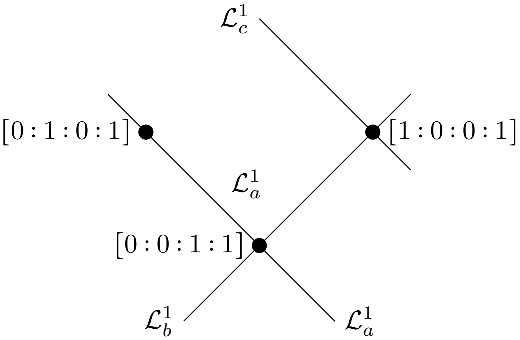

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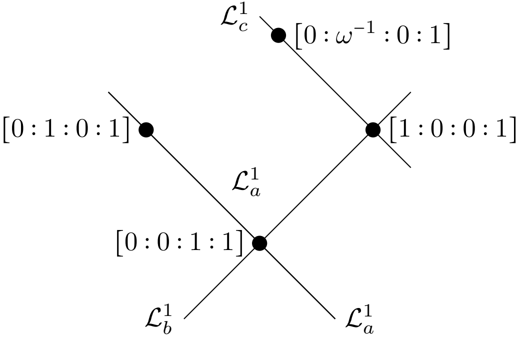

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Zariski tuple candidate

- Fundamental groups are almost always abelian (use Fermat curves and Kummer covers)
- The invariants are lost for $\mathscr{M}^{2d}_{3(d,d+1)}$.