

Arithmetic Zariski pairs of line arrangements

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Combinatorics and Topology

Definition

Combinatorics: $\mathcal{C} := (\mathcal{L}, \mathcal{P})$, \mathcal{L} finite set of *lines* and $\mathcal{P} \subset \{P \subset \mathcal{L} \mid \#P \geq 2\}$ finite set of *points* mimic arrangement of lines and multiple points.

Definition (Realization of \mathcal{C})

A line arrangement in \mathbb{P}^2 : $(\mathcal{A}, \{\text{multiple points}\}) \leftrightarrow (\mathcal{L}, \mathcal{P})$

Combinatorial objects

$$\blacktriangleright \mathbb{Z}^{\mathcal{L}} = \bigoplus_{L \in \mathcal{L}} \mathbb{Z}x_L, \quad \frac{\mathbb{Z}^{\mathcal{L}}}{\mathbb{Z} \left(\sum_{L \in \mathcal{L}} x_L \right)} =: H_1^{\mathcal{C}} \cong H_1(\mathbb{P}^2 \setminus \mathcal{A}; \mathbb{Z})$$

$$\blacktriangleright x_P = \sum_{P < L} x_L$$

$$\{x_L \wedge x_P \in H_1^{\mathcal{C}} \wedge H_1^{\mathcal{C}} \mid P < L\} = H_2^{\mathcal{C}} \cong H_2(\mathbb{P}^2 \setminus \mathcal{A}; \mathbb{Z})$$

$$\blacktriangleright H_{\mathcal{C}}^0 = \mathbb{Z}, \quad H_{\mathcal{C}}^j \cong H^j(\mathbb{P}^2 \setminus \mathcal{A}; \mathbb{Z}) \text{ dual of } H_j^{\mathcal{C}}, \quad j = 1, 2.$$



McLane arrangements I



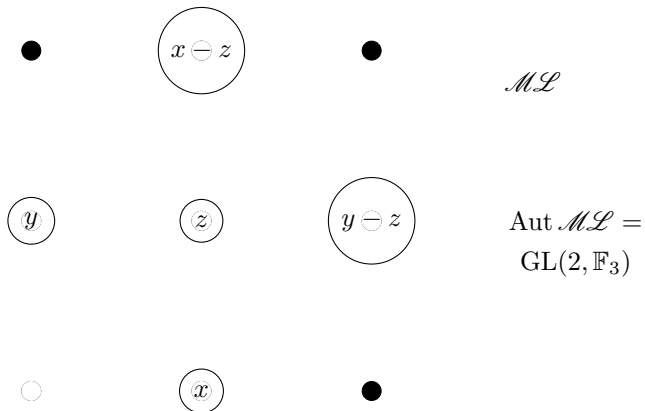
\mathcal{ML}



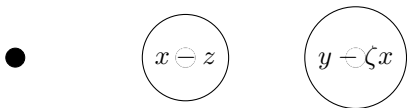
$\text{Aut } \mathcal{ML} =$
 $\text{GL}(2, \mathbb{F}_3)$



McLane arrangements I



McLane arrangements I



\mathcal{ML}



$\text{Aut } \mathcal{ML} =$
 $\text{GL}(2, \mathbb{F}_3)$



McLane arrangements I

$$(\zeta-1)x - y + z$$

$$x - z$$

$$y - \zeta x$$

$$\zeta^2 + \zeta + 1 = 0$$

\mathcal{ML}_\pm

$$y$$

$$z$$

$$y - z$$

$\text{Aut } \mathcal{ML} =$
 $\text{GL}(2, \mathbb{F}_3)$

$$0$$

$$x$$

$$x + (1 - \bar{\zeta})y - z$$



McLane arrangements II

Theorem (Rybnikov)

$\exists \varphi : \pi_1(\mathbb{P}^2 \setminus \mathcal{ML}_+) \rightarrow \pi_1(\mathbb{P}^2 \setminus \mathcal{ML}_-)$ group automorphism inducing the identity on homology.

Corollary

$\exists \rho : (\mathbb{P}^2, \mathcal{ML}_+) \rightarrow (\mathbb{P}^2, \mathcal{ML}_-)$ homeomorphism respecting orientations and ordering.

Orientation

$\exists \rho : (\mathbb{P}^2, \mathcal{ML}_+) \rightarrow (\mathbb{P}^2, \mathcal{ML}_-)$ homeomorphism respecting ordering and reversing orientation: *complex conjugation*.

Order

$\exists \rho : (\mathbb{P}^2, \mathcal{ML}_+) \rightarrow (\mathbb{P}^2, \mathcal{ML}_-)$ homeomorphism respecting orientation: $\mathrm{GL}(2, \mathbb{F}_3) \setminus \mathrm{SL}(2, \mathbb{F}_3)$.



Rybnikov's combinatorics

$\mathcal{RB} = \mathcal{ML}_1 \cup_{xz(x-z)=0} \mathcal{ML}_2$ (gluing in general position)

Theorem

$$G_{+++} = \pi_1(\mathbb{P}^2 \setminus \mathcal{RB}_{+++}) \not\cong \pi_1(\mathbb{P}^2 \setminus \mathcal{RB}_{+-}) = G_{--}$$

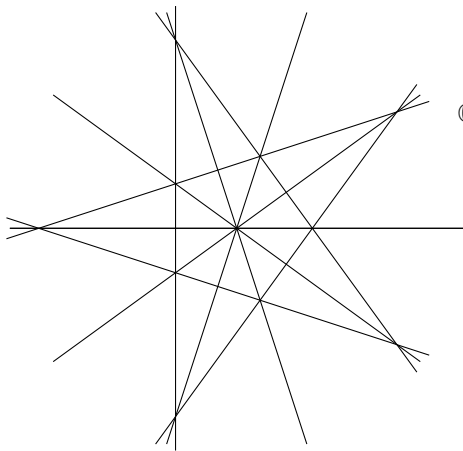
Guidelines of the proof.

Assume they are isomorphic $\implies G_{+++}/\gamma_4(G_{+++}) \cong G_{+-}/\gamma_4(G_{+-})$

1. The isomorphism induces the \pm identity on $H_1^{\mathcal{RB}}$ (purely combinatorial).
2. It does not happen using *truncated Alexander invariant*.



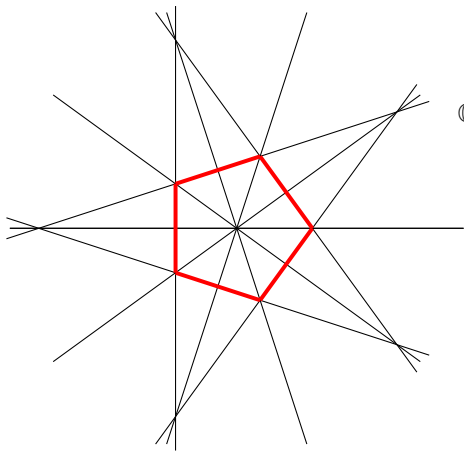
Pentagon and Pentagram



\mathcal{L}_0
 $\mathbb{Q}[\sqrt{5}]$



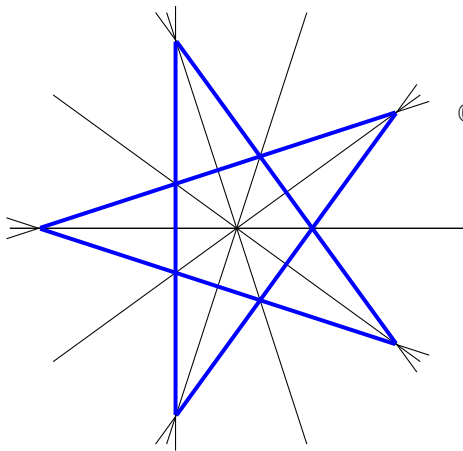
Pentagon and Pentagram



\mathcal{L}_0
 $\mathbb{Q}[\sqrt{5}]$



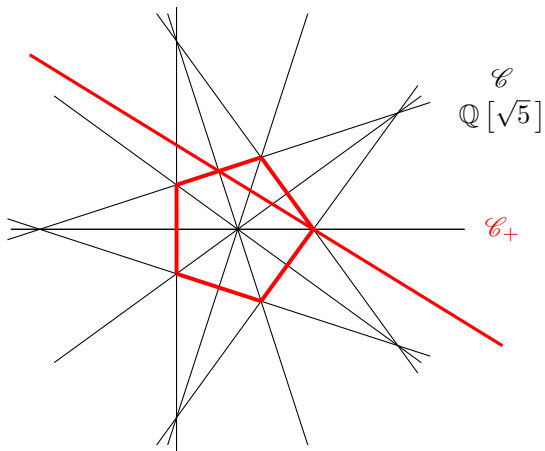
Pentagon and Pentagram



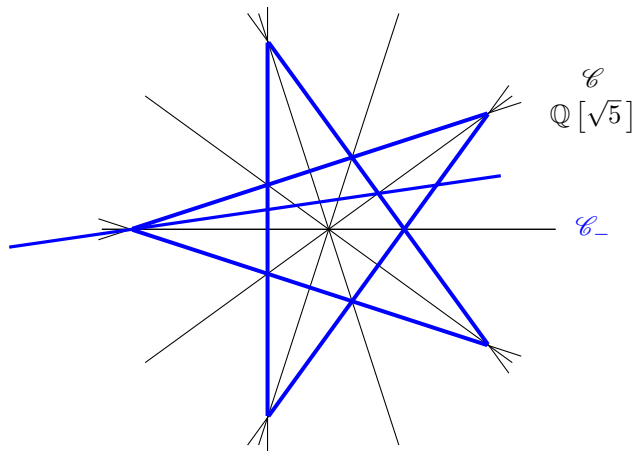
$$\mathcal{L}_0 \\ \mathbb{Q}[\sqrt{5}]$$



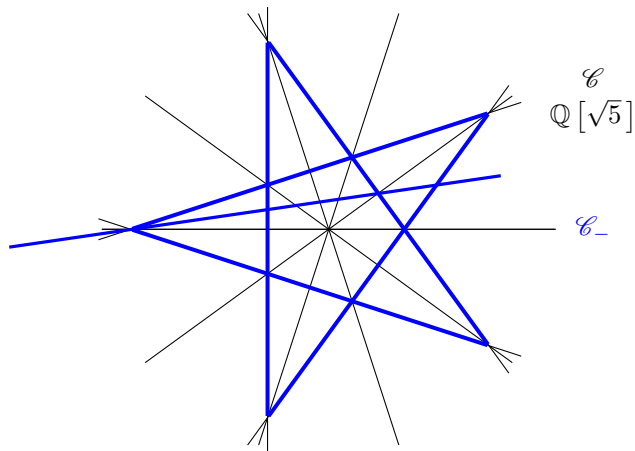
Pentagon and Pentagram



Pentagon and Pentagram



Pentagon and Pentagram



Theorem

There is no homeomorphism between $(\mathbb{P}^2, \mathcal{C}_+)$ and $(\mathbb{P}^2, \mathcal{C}_-)$

\mathcal{G}_{91} combinatorics

P_1



P_2



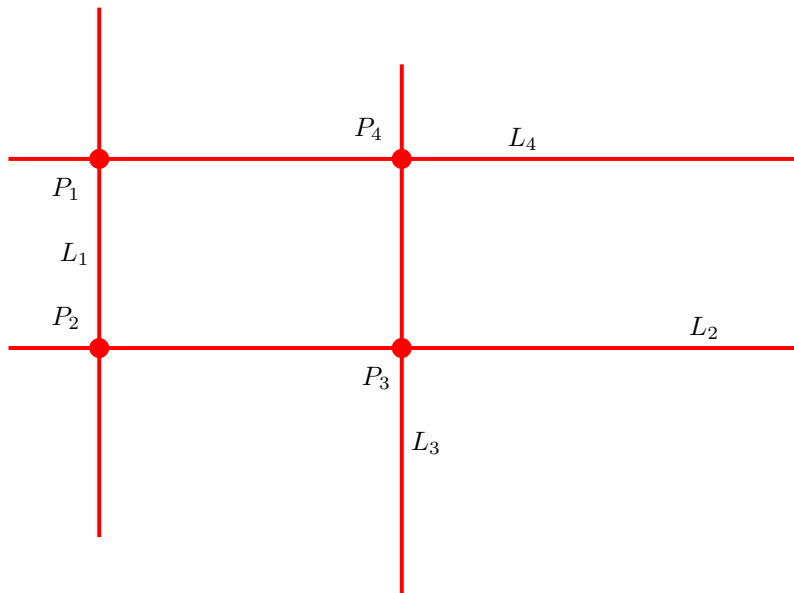
P_4



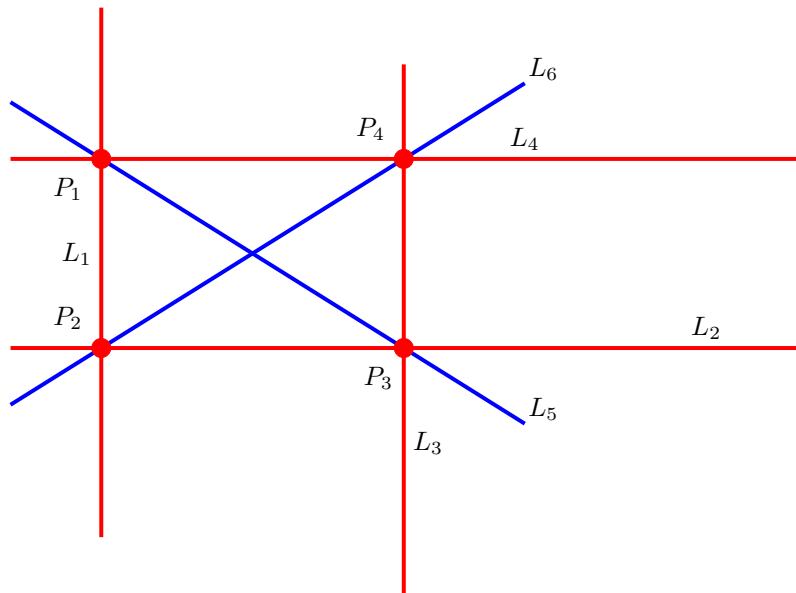
P_3



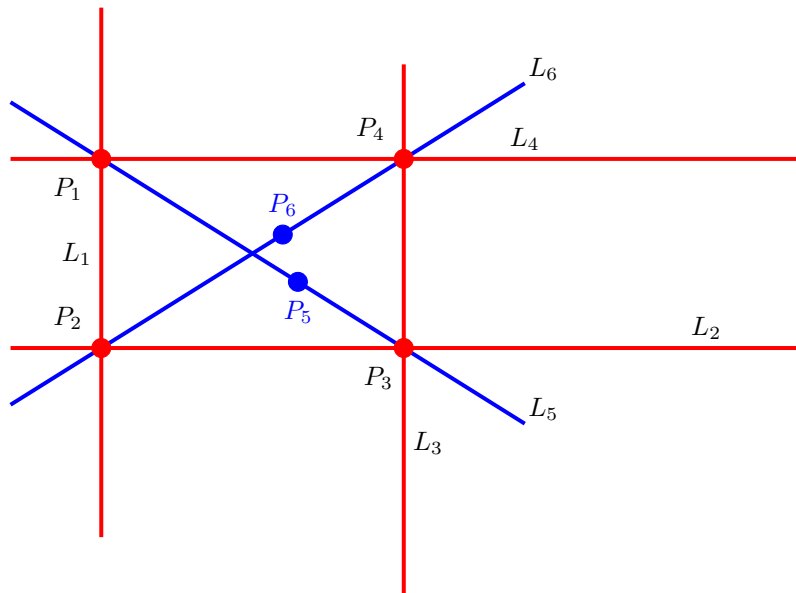
\mathcal{G}_1 combinatorics



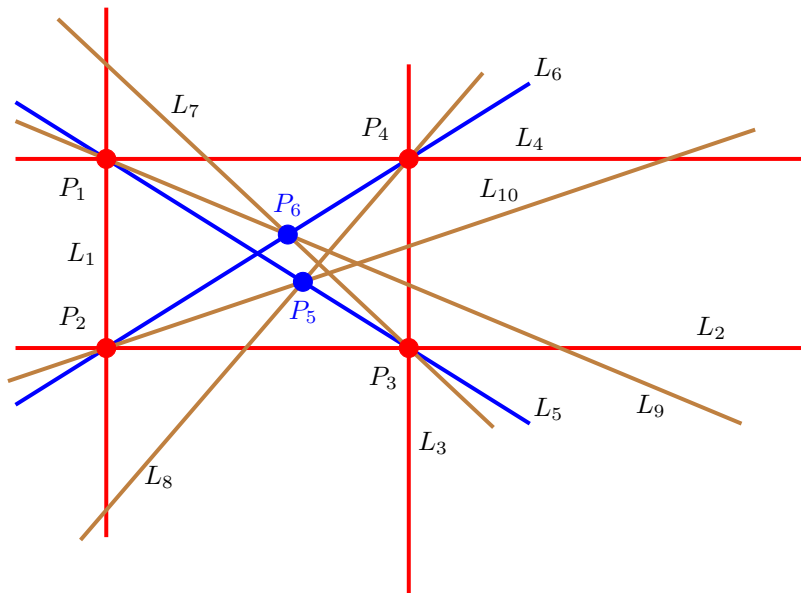
\mathcal{G}_{91} combinatorics



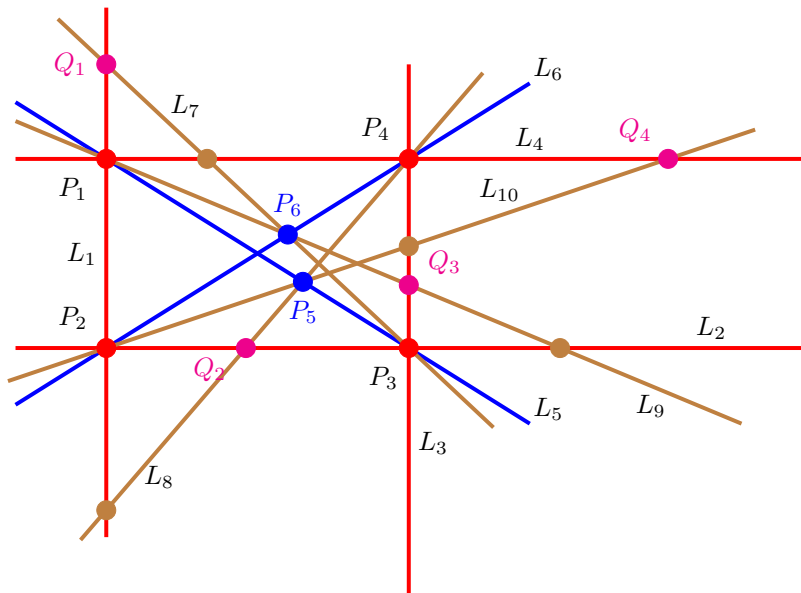
\mathcal{G}_{91} combinatorics



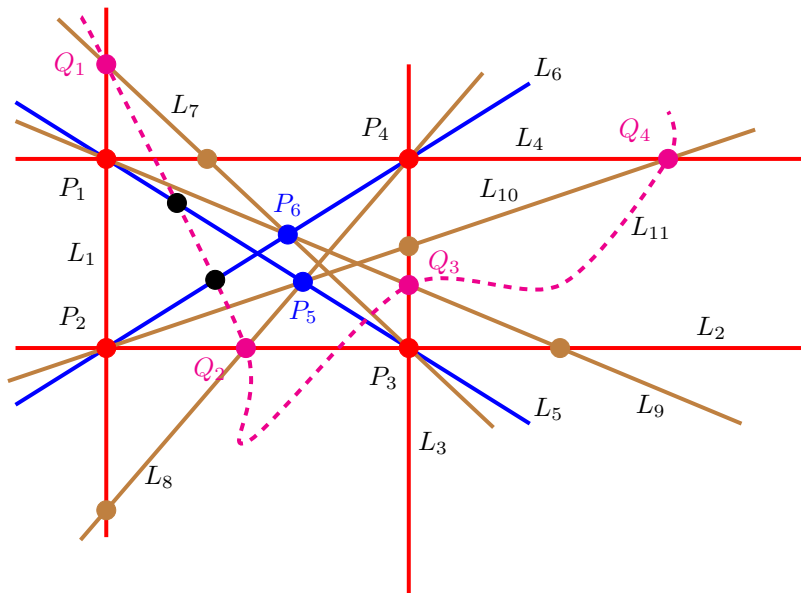
\mathcal{G}_{91} combinatorics



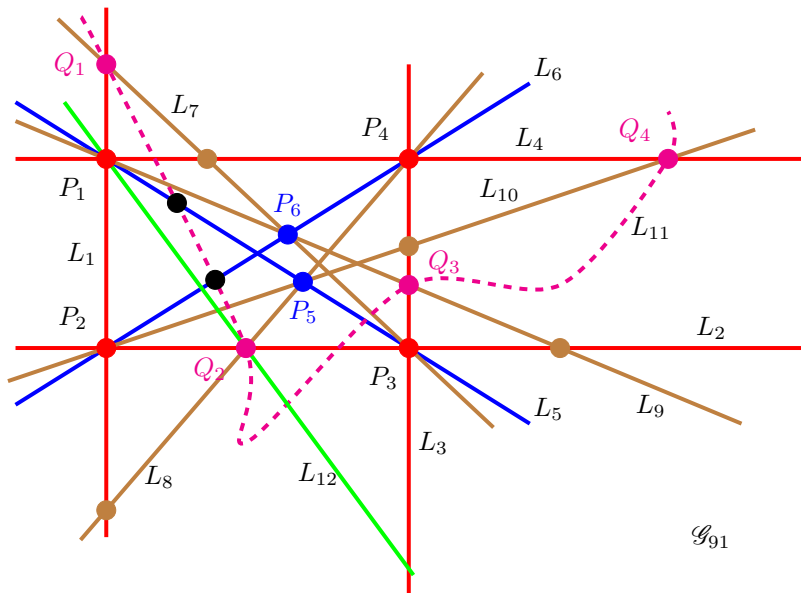
\mathcal{G}_{91} combinatorics



\mathcal{G}_{91} combinatorics



\mathcal{G}_{91} combinatorics



\mathcal{G}_{91}

Guerville's example

Theorem

\mathcal{G}_{91} admits four (Galois-conjugate) realizations \mathcal{A}_ζ with equations in the cyclotomic field \mathbb{K}_5 , for ζ a primitive fifth root of unity.

There is no oriented homeomorphism $(\mathbb{P}^2, \mathcal{A}_{\zeta_1}) \rightarrow (\mathbb{P}^2, \mathcal{A}_{\zeta_2})$ if $\zeta_1 \neq \zeta_2$.

Corollary

There is no homeomorphism $(\mathbb{P}^2, \mathcal{A}_\zeta) \rightarrow (\mathbb{P}^2, \mathcal{A}_{\zeta^2})$.

Proof.

Use linking number over a character of order 5. □



Main result I

Theorem

The groups $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_\zeta)$ and $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{\zeta^2})$ are not isomorphic (while their profinite completions are).

First step

$\varphi : \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_\zeta) \rightarrow \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{\zeta^2})$ isomorphism $\implies \varphi_* = \pm 1_{H_1^{\mathcal{G}_{91}}}$.

- ▶ Purely combinatorial statement.

Homological rigidity

- ▶ $\rho : H_1^{\mathcal{C}} \rightarrow H_1^{\mathcal{C}}$ is an *admissible isomorphism* if $(\rho \wedge \rho)(H_2^{\mathcal{C}}) = H_2^{\mathcal{C}}$
- ▶ $\mathcal{A}_1, \mathcal{A}_2$ realizations of \mathcal{C} , $\varphi : \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_1) \rightarrow \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_2)$ isomorphism $\implies \varphi_*$ admissible.
- ▶ \mathcal{C} *homologically rigid* if

$$\pm \text{Aut}(\mathcal{C}) = \{\text{admissible isomorphisms}\}$$

- ▶ ρ admissible $\implies \rho^* : H_{\mathcal{C}}^1 \rightarrow H_{\mathcal{C}}^1$ respects the resonance varieties.
- ▶ $\{H_S \text{ irreducible components of resonance varieties in } H^1\} \leftrightarrow \{S \text{ combinatorial pencil}\}$ (type point, type Ceva, type Hesse, ...)
- ▶ ρ^* sends *triangles* to *triangles*

Triangle

S_1, S_2, S_3 combinatorial pencils such that

$$\text{codim} \bigcap_i H_{S_i} = \sum_i \text{codim } H_{S_i} - 1.$$

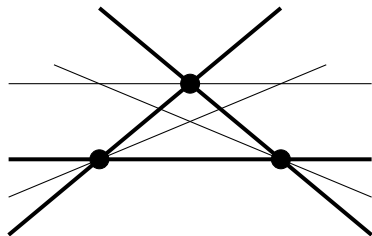


Homological rigidity

- ▶ $\rho : H_1^{\mathcal{C}} \rightarrow H_1^{\mathcal{C}}$ is an *admissible isomorphism* if $(\rho \wedge \rho)(H_2^{\mathcal{C}}) = H_2^{\mathcal{C}}$
- ▶ $\mathcal{A}_1, \mathcal{A}_2$ realizations of \mathcal{C} , $\varphi : \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_1) \rightarrow \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_2)$ isomorphism $\implies \varphi_*$ admissible.
- ▶ \mathcal{C} *homologically rigid* if

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- ▶ $\{H_S \text{ irreducible components of resonance varieties in } H^1\} \leftrightarrow \{S \text{ combinatorial pencil}\}$ (type point, type Ceva, type Hesse, ...)
- ▶ ρ^* sends *triangles* to *triangles*



Triangles in \mathcal{G}_1

i	S_i	$\dim H_S$	Δ_S	Δ_{S,P_1}
1	1, 7, 11	2	18	7
2	3, 9, 11	2	22	8
3	4, 10, 11	2	21	7
4	5, 8, 10	2	24	7
5	6, 9, 7	2	16	6
6	1, 2, 6, 10	3	53	12
7	2, 3, 5, 7	3	49	13
8	2, 8, 11, 12	3	57	15
9	4, 3, 6, 8	3	50	12
10	1, 4, 5, 9, 12	4	91	91
11	1, 2, 3, 4, 5, 6	2	24	8
12	1, 2, 4, 6, 8, 12	2	24	8
13	1, 2, 4, 10, 11, 12	2	20	7
14	1, 2, 5, 6, 7, 9	2	14	7
15	1, 2, 5, 7, 11, 12	2	14	7
16	1, 2, 5, 8, 10, 12	2	20	8
17	1, 3, 5, 7, 9, 11	2	14	7
18	1, 4, 5, 6, 8, 10	2	19	6
19	2, 3, 4, 5, 8, 12	2	20	8
20	2, 3, 5, 6, 8, 10	2	14	0
21	2, 3, 5, 9, 11, 12	2	18	9
22	2, 4, 6, 8, 10, 11	2	15	0
23	3, 4, 5, 6, 7, 9	2	12	6
24	3, 4, 8, 9, 11, 12	2	13	7
25	4, 5, 8, 10, 11, 12	2	15	7



Behind homological rigidity

Theorem (Marco)

- ▶ $P := (L_1, \dots, L_m)$ is a pencil (type point)
- ▶ It can be distinguished by triangles
- ▶ $\rho^* : H_{\mathcal{C}}^1 \rightarrow H_{\mathcal{C}}^1$ admissible

Then $\rho^*(H_P) = H_P$, where

$$H_P = \mathbb{Z}\langle x_{L_1}^* - x_{L_2}^*, \dots, x_{L_1}^* - x_{L_m}^* \rangle$$



Main result II

Theorem

The groups $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_\zeta)$ and $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{\zeta^2})$ are not isomorphic (while their profinite completions are).

First step

$\varphi : \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_\zeta) \rightarrow \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{\zeta^2})$ isomorphism $\implies \varphi_* = \pm 1_{H_1^{\mathcal{G}_{91}}}$.

Second step

There is no isomorphism such that $\varphi : \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_\zeta) \rightarrow \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{\zeta^2})$ isomorphism $\implies \varphi_* = 1_{H_1^{\mathcal{G}_{91}}}$

Truncated Alexander Invariant

- ▶ \mathcal{C} combinatorics, $\mathcal{A} = \{L_0, L_1, \dots, L_\ell\}$ realization
- ▶ $G_{\mathcal{A}} := \pi_1(\mathbb{P}^2 \setminus \mathcal{A})$, $\Lambda := \mathbb{Z}[H_1^{\mathcal{C}}] = \mathbb{Z}[t_1^{\pm 1}, \dots, t_\ell^{\pm 1}]$,
 $t_0 = (t_1 \cdots t_\ell)^{-1}$
- ▶ $M_{\mathcal{A}} := G'_{\mathcal{A}}/G''_{\mathcal{A}}$ as Λ -module is the *Alexander invariant*.
- ▶ $\mathfrak{m} \subset \Lambda$ augmentation ideal of Λ :

$$\ker(\Lambda \rightarrow \mathbb{Z}), \quad \prod_{j=0}^{\ell} t_j^{n_j} \in H_1^{\mathcal{C}} \mapsto 1$$

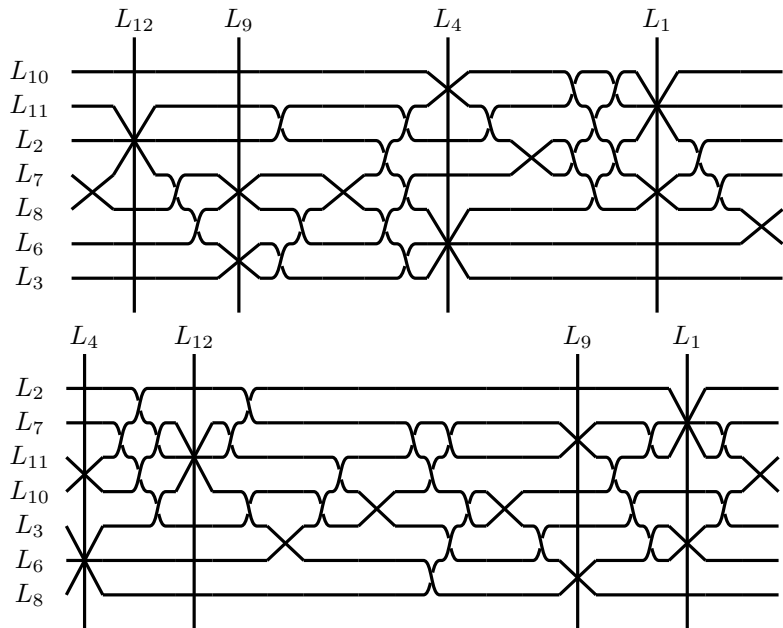
- ▶ $M_{\mathcal{A}}^k := M/\mathfrak{m}^k M = M \otimes_{\Lambda} \Lambda/\mathfrak{m}^k$ *truncated Alexander invariant*.
- ▶ Denote $s_i := t_i - 1 \in \mathfrak{m}$.
- ▶ $\theta_{k+2}(G_{\mathcal{A}}) := \ker(\varphi_k : G'_{\mathcal{A}} \rightarrow M_{\mathcal{A}}^k)$ Chen group



Truncated Alexander Invariant

- ▶ $G_{\mathcal{A}} = \langle x_1, \dots, x_\ell \mid R_1, \dots, R_s \rangle$,
- ▶ $M_{\mathcal{A}}$ generated by $x_{i,j} = [x_i, x_j] \bmod G''$ and relators:
 - ▶ Rewriting R_j
 - ▶ $[x_i, x_j] = 1 \mapsto x_{i,j} = 0$
 - ▶ $[x_i, x_j x_k] = 1 \mapsto 0 = x_{i,j} + t_j \cdot x_{i,k} = x_{i,j} + x_{i,k} + s_j \cdot x_{i,k}$
 - ▶ $[x_i, x_k x_j x_k^{-1}] = 1 \mapsto s_i \cdot x_{k,j} + x_{i,j} = 0$
 - ▶ Jacobi relations: $s_i \cdot x_{j,k} + s_j \cdot x_{k,i} + s_k \cdot x_{i,j} = 0$
- ▶ $\text{gr}^k M_{\mathcal{A}}$, $k = 0, 1$, is combinatorial.
 - ▶ $\text{gr}^0 M_{\mathcal{A}} = (H_1^{\mathcal{C}} \wedge H_1^{\mathcal{C}}) / H_2^{\mathcal{C}}$ generated by $x_{i,j}$
 - ▶ $\text{gr}^1 M_{\mathcal{A}} \subset M_{\mathcal{A}}^2$ generated by $s_i \cdot x_{j,k}$
 - ▶ $0 \rightarrow \text{gr}^1 M_{\mathcal{A}} \rightarrow M_{\mathcal{A}}^2 \rightarrow \text{gr}^0 M_{\mathcal{A}} \rightarrow 0$ splits non canonically.
- ▶ $g \in H_1$ and $p \in M_{\mathcal{A}}^k \implies [g, p] \in M_{\mathcal{A}}^{k+1}$.

Wiring diagrams



Steps of the proof

- ▶ Assume an isomorphism $\varphi : G_{\mathcal{A}_\zeta} \rightarrow G_{\mathcal{A}_{\zeta^2}}$, $x'_i \mapsto x''_i \cdot g_i$, $g_i \in G'_{\mathcal{A}_{\zeta^2}}$
- ▶ $\text{rk } M_{\mathcal{A}_\zeta}^1 = \text{rk } M_{\mathcal{A}_{\zeta^2}}^1 = \text{rk } \text{gr}^0 M_2^{\mathcal{G}_{91}} = 23$, basis $\{x_{i,j} \mid (i,j) \in \mathcal{B}\}$.
- ▶ $\varphi_* : M_{\mathcal{A}_\zeta}^2 \rightarrow M_{\mathcal{A}_{\zeta^2}}^2$. Need:

$$g_i \equiv \sum_{(j,k) \in \mathcal{B}} n_{i,j,k} x''_{j,k} \in M_{\mathcal{A}_{\zeta^2}}^1, \quad n_{i,j,k} \in \mathbb{Z}$$

$$x'_{i,j} \mapsto [x''_i \cdot g_i, x''_j \cdot g_j] = x''_{i,j} + s_i \cdot g_j - s_j \cdot g_i$$

- ▶ R_i , $i = 1, \dots, 32$, relation of $G_{\mathcal{A}_\zeta}$ rewritten in $M_{\mathcal{A}_\zeta}^2$. For example if L_i, L_j, L_k triple point:

$$x'_{i,j} + x'_{i,k} + \text{terms in } \text{gr}^1 M_{\mathcal{A}_\zeta}; \quad x'_{i,k} + x'_{j,k} + \text{terms in } \text{gr}^1 M_{\mathcal{A}_\zeta}$$

- ▶ $\varphi_*(R_i) \in M_{\mathcal{A}_{\zeta^2}}^2 \otimes \mathbb{Z}[n_{i,j,k}]$, more precisely

$$\varphi_*(R_i) \in \text{gr}^1 M_{\mathcal{A}_{\zeta^2}} \otimes \mathbb{Z}[n_{i,j,k}], \quad \text{rk } \text{gr}^1 M_2^{\mathcal{G}_{91}} \cong \mathbb{Z}^{91}$$



Linear equations

- ▶ We obtain a system \mathcal{S} of $32 \times 91 = 2912$ linear equations in $11 \times 23 = 253$ unknowns
- ▶ Existence of $\varphi \implies$ existence of integer solutions in \mathcal{S} .
- ▶ Solve \mathcal{S} with **Sagemath**.
- ▶ Solution over \mathbb{Q} : \mathbb{Q} -affine space of $\dim = 12$
- ▶ Smallest ring where \mathcal{S} admit solutions is $\mathbb{Z} \left[\frac{1}{5} \right]$.
- ▶ Whole process 314.02s CPU time (mostly for rewriting relations!).



Main result III

Theorem

The groups $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_\zeta)$ and $\pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{\zeta^2})$ are not isomorphic (while their profinite completions are).

First step

$\varphi : \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_\zeta) \rightarrow \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{\zeta^2})$ isomorphism $\implies \varphi_* = \pm 1_{H_1^{\mathcal{G}_{91}}}$.

Second step

There is no isomorphism such that $\varphi : \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_\zeta) \rightarrow \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{\zeta^2})$ isomorphism $\implies \varphi_* = 1_{H_1^{\mathcal{G}_{91}}}$

Third step

There is no isomorphism such that $\varphi : \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_\zeta) \rightarrow \pi_1(\mathbb{P}^2 \setminus \mathcal{A}_{\zeta^3})$ isomorphism $\implies \varphi_* = 1_{H_1^{\mathcal{G}_{91}}}$



Thank you
どうもありがとう

