Computational Methods in the Topology of Algebraic Varieties

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EACA
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- Proved in `arXiv:1511.09254`, using free resolution of ideals and properties of singular plane curves.

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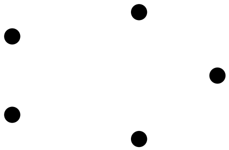
  - `Sagemath` development by Marco-Buzunáriz

From pentagons to line arrangements



$\mathbb{Q}(\sqrt{5})$

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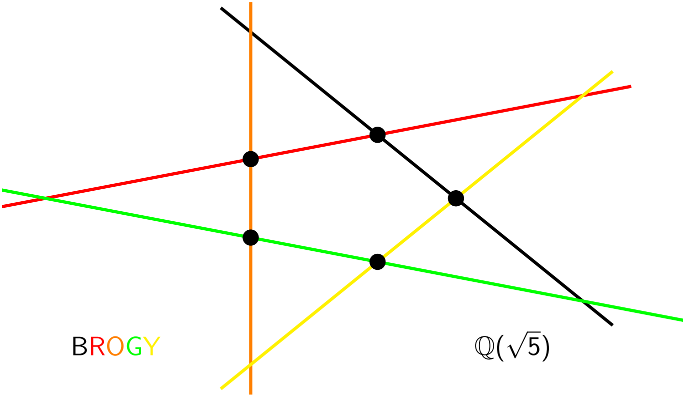

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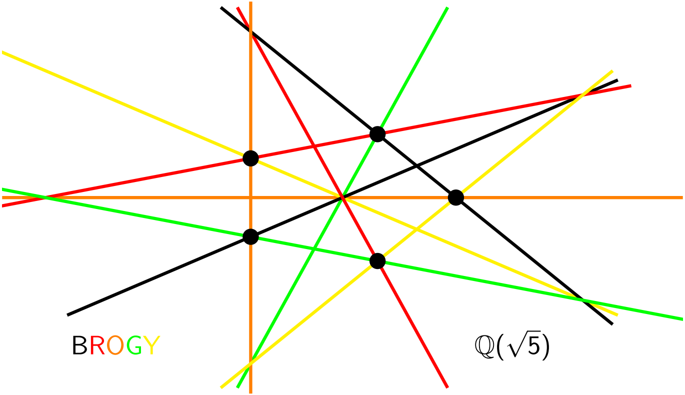

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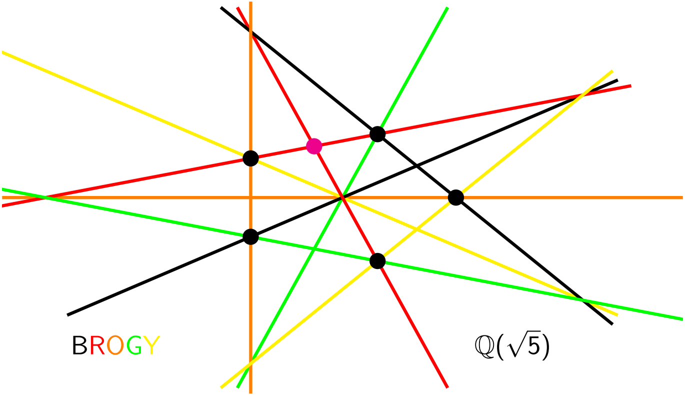

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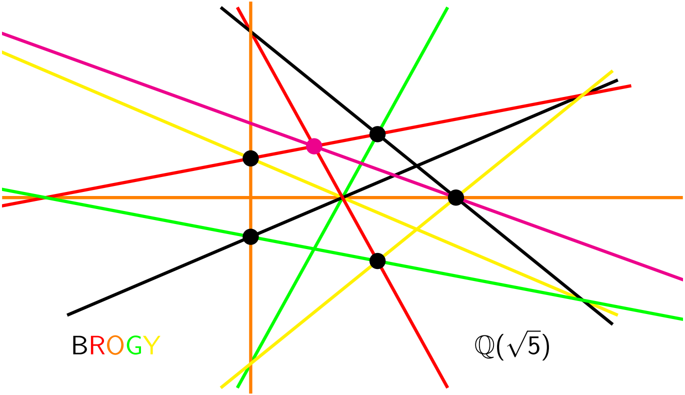
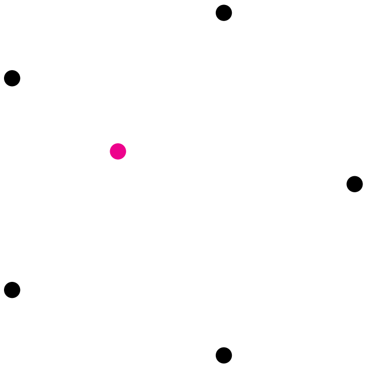


Conjugate arrangement



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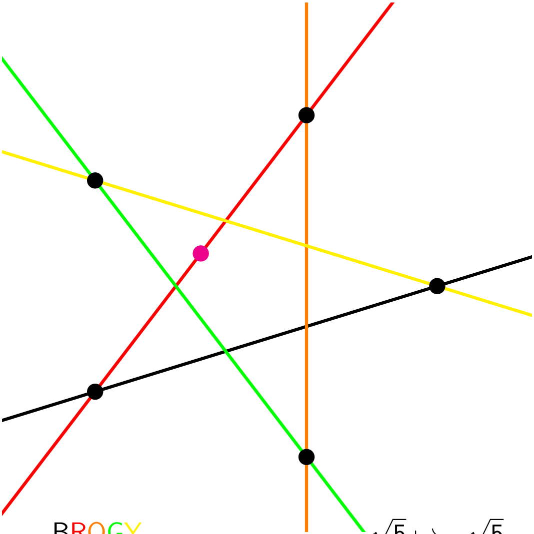

BROGY

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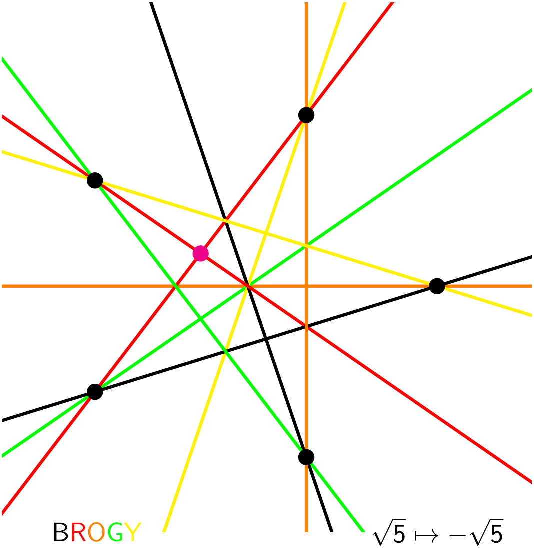

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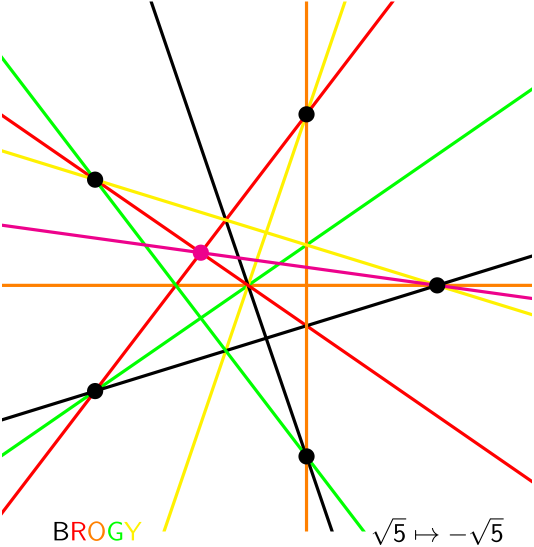
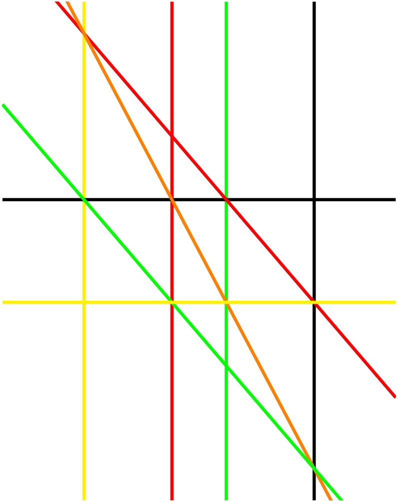


Vertical versions



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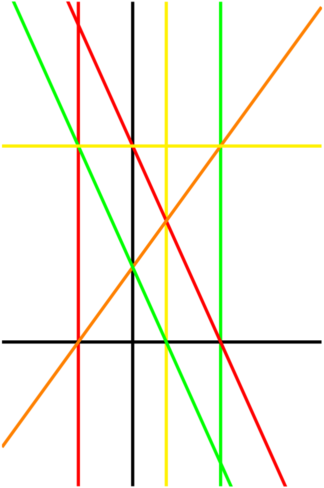

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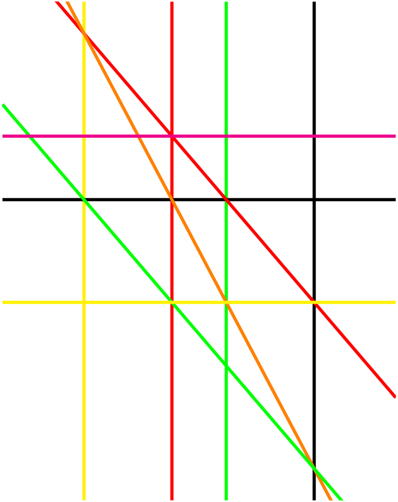

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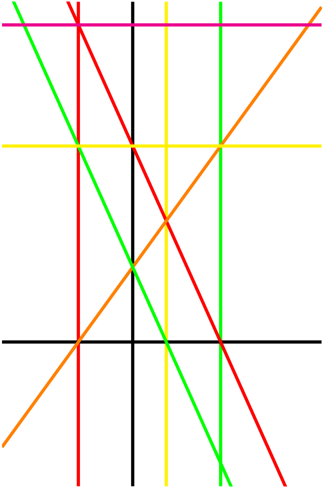

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Braid Monodromy and Topology

Theorem (_, Carmona, Cogolludo, Marco)

There is no homeomorphism $\Phi : (\mathbb{P}^2, \mathcal{A}_{\sqrt{5}}) \to (\mathbb{P}^2, \mathcal{A}_{-\sqrt{5}})$, $\mathbb{P}^2 = \mathbb{P}^2(\mathbb{C})$.

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- $G_\pm$ have the same finite quotients.

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- $C([g_1, \ldots, g_r]) = \langle g_1, \ldots, g_r \rangle$: monodromy group. Invariants of $G^r / \mathbb{B}_r$.

Hurwitz moves and braid monodromy

- Braid monodromy of an affine curve of degree d (without vertical asymptotes) with r non-transversal vertical lines: a Hurwitz class in $\mathbb{B}_d \setminus \mathbb{B}_d^r / \mathbb{B}_r$.

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- $\#\rho(\text{Pure braids}) = 58{,}032 \times 10^6$, centralizer of pseudo-Coxeter element has 115,200 elements, monodromy groups of order 30,000: no compatible conjugation.

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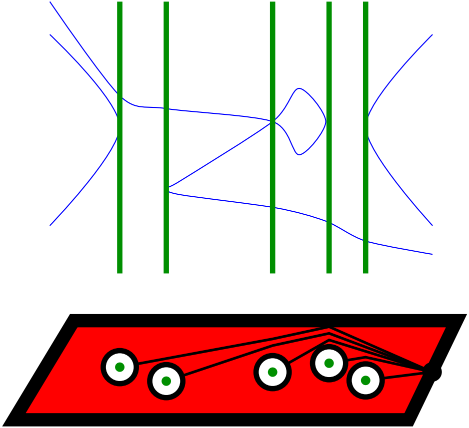

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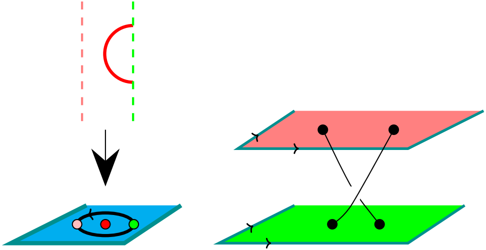

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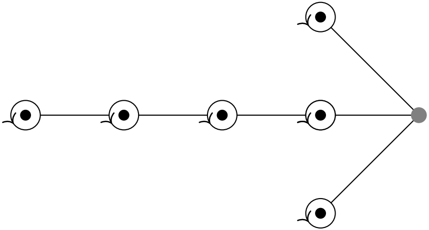

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- Zariski-van Kampen method: Bessis, Carmona, Berna and Amorós, Marco and Rodríguez.

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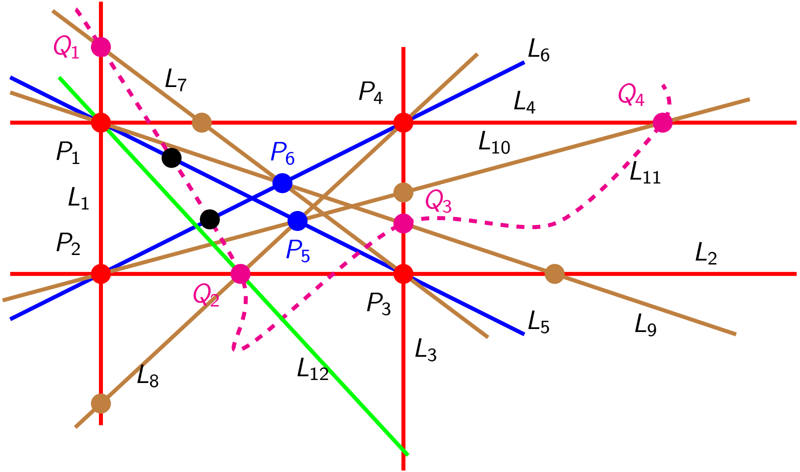

Another arithmetic Zariski pair I

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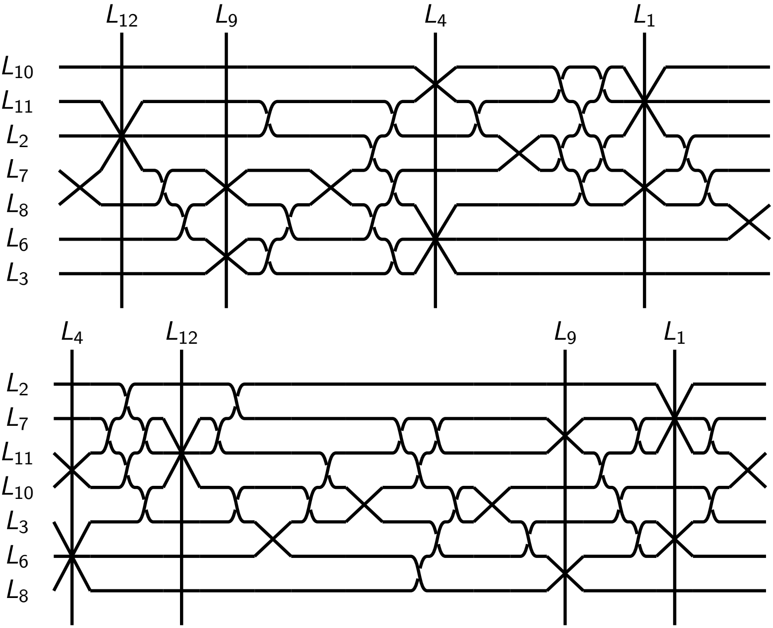
Compute the fundamental groups.

Another arithmetic Zariski pair II



Another arithmetic Zariski pair II

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Links

`https://github.com/enriqueartal/ZariskiPair12Lines.git`

Thank you