

Moduli spaces of plane hyperelliptic curves and embedded topology

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Definition

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$(X, \mathcal{C}, A)$ **embedded hyperelliptic curve**:

- ▶ $X = \mathbb{P}^2$
- ▶ $\mathcal{C} \subset \mathbb{P}^2$ irreducible curve of degree $d \geq 2$
- ▶ $A = P = [0 : 1 : 0] \in \mathbb{P}^2$
- ▶ $(C \setminus \{P\}) \cdot L = 2$, L general line through P , i.e., $m_P(C) = d - 2$.



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$(X, \mathcal{C}, A)$ **embedded hyperelliptic curve**:

- ▶ $X = \Sigma_1$ (blown-up of $\mathbb{P}^2$ at P)
- ▶ $A = E_1$ the exceptional component
- ▶ $\mathcal{C}$ irreducible curve linearly equivalent to $2(E_1 + \alpha F)$, $\alpha \geq 1$, F the class of a fiber.



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$(X, \mathcal{C}, A)$ **embedded hyperelliptic curve**:

- ▶ $X = \mathbb{P}^2_\omega$, $\omega = (1, n, 1)$, $n \geq 1$
- ▶ $\mathcal{C} \subset \mathbb{P}^2$ irreducible curve
- ▶ $A = P = [0 : 1 : 0]_\omega$
- ▶ $(C \setminus \{P\}) \cdot L = 2$, L is a general *line* through P .



Definition

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$(X, \mathcal{C}, A)$ **embedded hyperelliptic curve**:

- ▶ $X = \Sigma_n$ $n \geq 1$
- ▶ $A = E_n$ the negative section
- ▶ $\mathcal{C}$ irreducible curve linearly equivalent to $2(E_n + \alpha F)$, $\alpha \geq n$, F the class of a fiber.



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- ▶ $X = \mathbb{P}^2$
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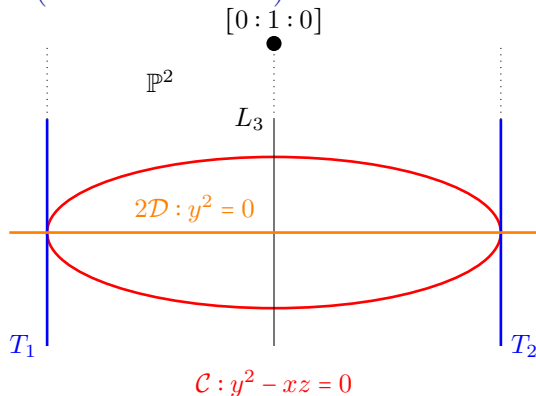
Goal

Given a fixed *combinatorial type* (degree and topological type of singularities):

- ▶ Study the moduli space $\mathcal{M}$ of such curves.
- ▶ Given an irreducible component of $\mathcal{M}_1 \subset \mathcal{M}$, study the embedded topological type of $(\mathbb{P}^2, \mathcal{C})$ of $\mathcal{C} \in \mathcal{M}_1$.



Toy example (trivial moduli)



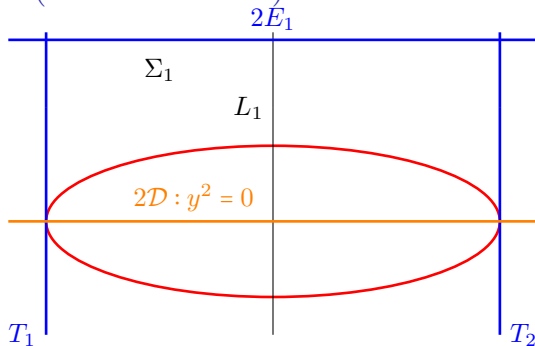
Pencil of lines through P : L_1 generic, T_1, T_2 special wrt $\mathcal{C}$.

Pencil

- ▶ $\mathcal{F}_1 : 2\mathcal{D}$
- ▶ $\mathcal{F}_2 : T_1 + T_2$
- ▶ $\mathcal{F}_3 : \mathcal{C}$



Toy example (trivial moduli)



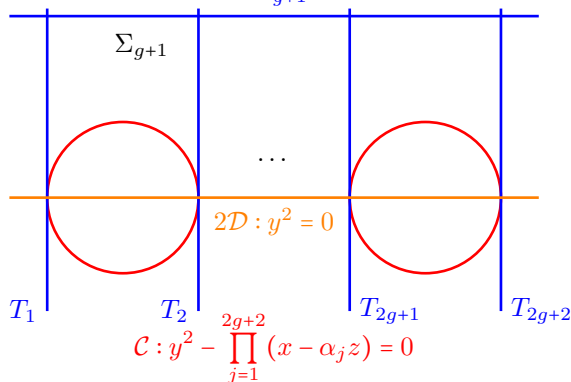
Pencil of fibers: L_1 generic, T_1, T_2 special wrt $\mathcal{C}$.

Pencil

- ▶ $\mathcal{F}_1 : 2\mathcal{D}$
- ▶ $\mathcal{F}_2 : T_1 + T_2 + 2E_1$
- ▶ $\mathcal{F}_3 : \mathcal{C}$



Generalized toy example (NON trivial moduli)



Pencil of fibers P : $T_1, \dots, T_{2g+2}$ special wrt $\mathcal{C}$.

Pencils

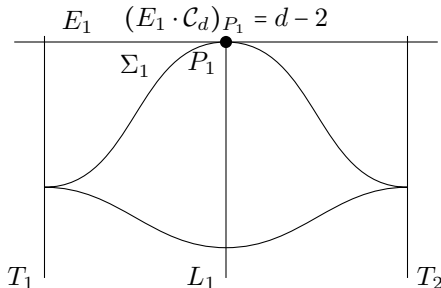
- ▶ $\mathcal{F}_1 : 2\mathcal{D}$
- ▶ $\mathcal{F}_2 : 2E_{g+1} + \sum_{j=1}^{2g+2} T_j$
- ▶ $\mathcal{F}_3 : \mathcal{C}$

Flenner–Zăidenberg curves, FZ1996

- ▶ Infinite family of rational cuspidal curves $\mathcal{C}_d$, $\# \text{Sing } \mathcal{C}_d = 3$.
- ▶ $(\mathcal{C}_d, v^{d-2} = u^{d-1})$, $(\mathcal{C}_d, v^2 = u^{2a+1})$, $(\mathcal{C}_d, v^2 = u^{2b+1})$.
- ▶ $d - 2 = a + b$, trivial moduli spaces.

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- ▶ $d - 2 = a + b$, trivial moduli spaces.

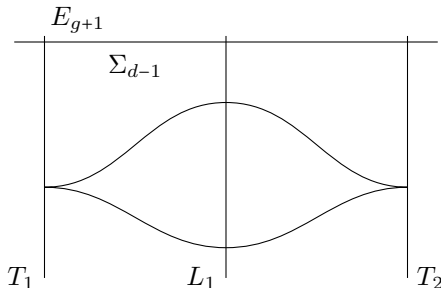


- ▶ Blow-up P_1 and blow-down the strict transform of L_3 . Repeat $d - 3$ more times.



Flenner–Zăidenberg curves, FZ1996

- ▶ $d - 2 = a + b$, trivial moduli spaces.

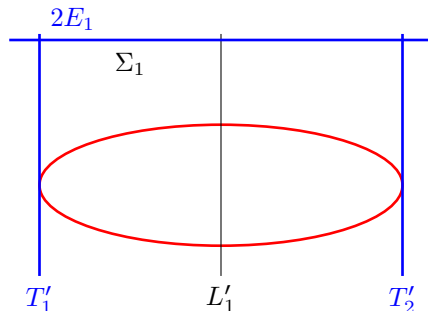


- ▶ Blow the cusps and blow down the strict transforms. Repeat $a - 1$ and $b - 1$ more times



Flenner–Zăidenberg curves, FZ1996

- ▶ $d - 2 = a + b$, trivial moduli spaces.



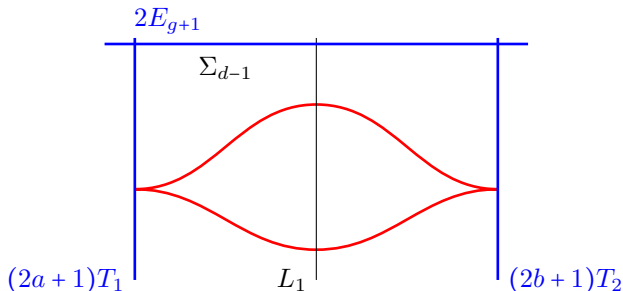
- ▶ Toy example! Come back and follow the pencil

$$\mathcal{F}_1 : 2\mathcal{D} \qquad \mathcal{F}_2 : T'_1 + T'_2 + 2E_1 \qquad \mathcal{F}_3 : \mathcal{C}$$



Flenner–Zaidenberg curves, FZ1996

- ▶ $d - 2 = a + b$, trivial moduli spaces.

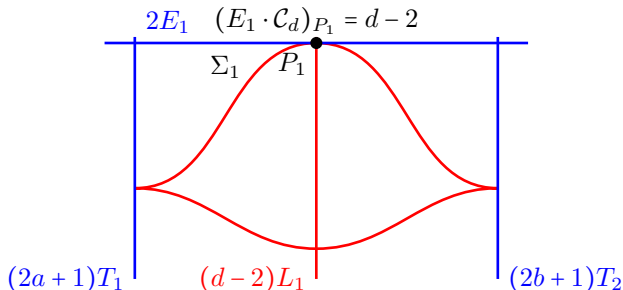


- ▶ $\mathcal{D}$ smooth and maximal contact at the cusps.

$$\mathcal{F}_1 : 2\mathcal{D} \quad \mathcal{F}_2 : (2a+1)T_1 + (2b+1)T_2 + 2E_{d-1} \quad \mathcal{F}_3 : \mathcal{C}$$

Flenner–Zaidenberg curves, FZ1996

- ▶ $d - 2 = a + b$, trivial moduli spaces.



- ▶ $\mathcal{D}$ curve of degree $d - 1$

$$\mathcal{F}_1 : 2\mathcal{D} \quad \mathcal{F}_2 : (2a+1)T_1 + (2b+1)T_2 + 2E_1 \quad \mathcal{F}_3 : \mathcal{C} + (d-2)L_1$$



Flenner–Zaidenberg curves, FZ1996

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Lemma

There is a surjective $\exists \Phi : \mathbb{P}^2 \setminus \mathcal{C}_d \twoheadrightarrow \mathbb{P}^{1,\text{orb}}$ orbifold morphism with connected fibers where the orbifold points are

- ▶ $\Phi(\mathcal{D})$ of index 2;
- ▶ $\Phi(L_1) = \Phi(L_2)$ of index $\gcd(2a + 1, 2b + 1) =: 2c + 1$ (odd!);
- ▶ $\Phi(L_3)$ of index $d - 2$.

In particular, $\Phi_* : \pi_1(\mathbb{P}^2 \setminus \mathcal{C}_d) \twoheadrightarrow \pi_1^{\text{orb}}(\mathbb{P}_{(2,2c+1,d-2)}^1)$



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A fundamental group

Using a braid monodromy in $\mathbb{B}_2 \equiv \mathbb{Z}$, $\pi_1(\mathbb{P}^2 \setminus (\mathcal{C} \cup T'_1 \cup T'_2 \cup L'_1))$ is

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$$\langle x, y, e, \tau_1, \tau_2, \ell \mid \tau_j x \tau_j^{-1} = y, [x, \tau_j x \tau_j] = 1, [x, \ell] = [y, \ell] = xye = 1, \tau_1 \tau_2 \ell = e \rangle$$



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Lemma

The meridians of the lines T_1, T_2, T_3 for $\mathcal{C}_d$ are: $\tau_1 e^a, \tau_2 e^b, \ell x^{d-2}$.

Killing these meridians we obtain $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_d)$, generated by $\boxed{x, y}$ with relations:



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- ▶ $(xy)^a x = y(xy)^a, (xy)^b x = y(xy)^b \implies (xy)^c x = y(xy)^c,$
- ▶ x^{d-2} central,
- ▶ $(xy)^{a+b+1} = x^{d-2} \implies (xy)^{r(2c+1)} = x^{d-2},$ since $2c+1$ divides $(2a+1) + (2b+1) = 2(a+b+1) = 2(d-1)$

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Killing these meridians we obtain $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_d)$, generated by $\boxed{x, y}$ with relations:

$$(xy)^c x = y(xy)^c, (xy)^{r(2c+1)} = x^{d-2}, \quad d-1 = r(2c+1).$$

$$A = xy, \quad B = (xy)^c x, \quad C = x, \quad T = B^2$$

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$$\begin{aligned}\pi_1(\mathbb{P}^2 \setminus C) &= \langle A, B, C, T \mid T = A^{2c+1} = B^2 = BA^cC, C^{d-2} = T^r \rangle \\ &\underset{D=A^c}{\cong} \langle B, C, D, T \mid T = B^2 = BDC, C^{d-2} = T^r, D^{2c+1} = T^c \rangle\end{aligned}$$

We recover the epimorphism on the orbifold group.

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We recover the epimorphism on the orbifold group.

Theorem (A_1997)

$(2, d-2, 2c+1)$ be the orbifold multiplicities of $\mathcal{C}_d$, Flenner-Zaĭdenberg curve.

- ▶ If one value is 1 the fundamental group is abelian.
- ▶ The fundamental group is finite non-abelian *only* for $(2, 3, 2)$ (triscuspidal quartic, [Zariski1929]) and $(2, 5, 3)$.
- ▶ For any other case the fundamental group is infinite (non-abelian).



Strategy

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A hyperelliptic curve $(X, \mathcal{C}, A)$ is **strict** if $\mathcal{C} \cap A = \emptyset$. It is **strict smooth** if it is smooth and strict.



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Fact

Strict smooth $\equiv$ generalized toy example.



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Definition ($n > 0$)

A **negative Nagata transformation** of Σ_n is a birational map $\Sigma_n \dashrightarrow \Sigma_{n+1}$:

- ▶ Blow up $Q \in E_n$, F_Q its fiber,
- ▶ Blow down the strict transform of F_Q



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Definition ($n > 0$)

A **positive Nagata transformation** of Σ_n is a birational map $\Sigma_n \dashrightarrow \Sigma_{n-1}$:

- ▶ Blow up $Q \notin E_n$, F_Q its fiber,
- ▶ Blow down the strict transform of F_Q



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Lemma

$A, B \subset \Sigma_n$ irreducible, $n \geq 1$; $A \cap B = \emptyset$ if and only if

- ▶ $A, B \sim F$
- ▶ $A = E_n$, $B \sim k(E_n + nF)$



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$(\mathbb{P}^2, \mathcal{C}_d, P)$ hyperelliptic curve of genus g

- Blow-up P : $(\Sigma_1, \mathcal{C}_d, E_1)$, $\mathcal{C}_d \cap E_1 = \{P_1, \dots, P_n\}$ (corresponding to the tangent directions of $(\mathcal{C}_d, P)$).



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- ▶ A series of negative Nagata transformations at $P_1, \dots, P_n$ and infinitely near points:

$$(\Sigma_m, \widehat{\mathcal{C}}_d, E_m), m \geq 1, \text{strict}, \text{Sing } \widehat{\mathcal{C}}_d = \{Q_1, \dots, Q_h\}, (\widehat{\mathcal{C}}_d, Q_j) \cong \mathbb{A}_{n_i}$$



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- ▶ A series of positive Nagata transformations at $Q_1, \dots, Q_h$ and infinitely near points: $(\Sigma_{g+1}, \widetilde{\mathcal{C}}_d, E_{g+1})$, strict smooth (possible using the properties of disjoint curves).

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- ▶ Some fibers may support both positive and negative transformations.

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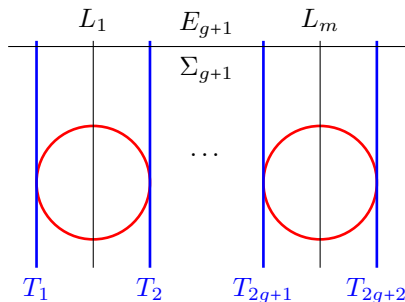
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Fundamental group with lines

$$\pi_1(\Sigma_{g+1}) \setminus (T_1 \cup \cdots \cup T_{2g+2} \cup L_1 \cdots \cup L_m)$$



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Fundamental group with lines

$\pi_1(\Sigma_{g+1}) \setminus (T_1 \cup \cdots \cup T_{2g+2} \cup L_1 \cdots \cup L_m)$ is generated by $x, y, e, \tau_1, \dots, \tau_{2g+2}, \ell_1, \dots, \ell_m$, with relations

$$1 = xye = [\ell_j, x] = [\ell_j, y] = [x, \tau_i x \tau_i], \tau_i y \tau_i^{-1} = x,$$

and

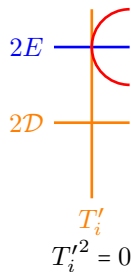
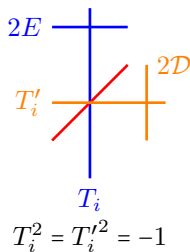
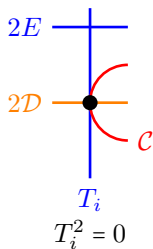
$$\prod_{i=1}^{2g+2} \tau_i \prod_{j=1}^m \ell_j = e^{g+1}.$$

To compute $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_d)$, we compute (and kill) the meridians of the new lines $T'_1, \dots, T'_{2g+2}, L'_1, \dots, L'_m$ after the Nagata transformations.



Nagata transformations

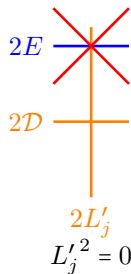
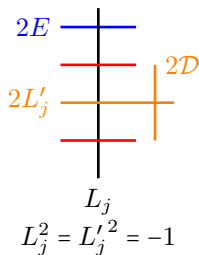
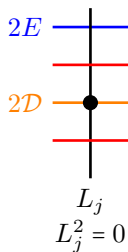
- Positive Nagata transformation at a tangency point and general positive transformations



- Fiber: $2\mathcal{D} + (1 + 2c_i)T'_i + \dots$
- Meridian: $x\tau_i$
- On π_1 : $x = y$

Nagata transformations

- Positive Nagata transformation at a general point in L_j and general positive transformations

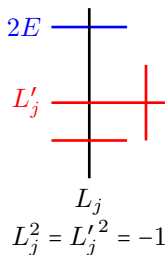
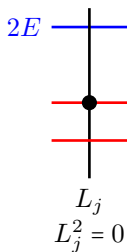


Topologically we may assume the point is in $\mathcal{D}$

- Fiber: $2\mathcal{D} + 2c'_j L'_j + \dots$
- Meridian: ℓ_j
- On π_1 : $\ell_j = 1$.

Nagata transformations

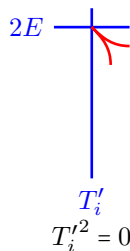
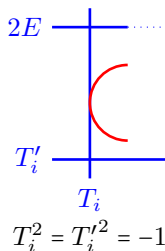
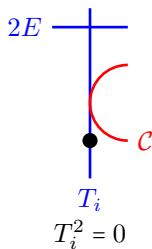
- Positive Nagata transformations at a point in $L_j \cap \mathcal{C}$ (q_j times) and general positive transformations



- ▶ Fiber: $\mathcal{C} + q_j L'_j + \dots$
- ▶ Meridian: $x^{q_j} \ell_j$
- ▶ On π_1 : $[x^{q_j}, y] = 1$

Nagata transformations

- ▶ Positive Nagata transformations at a general point in $T_i \cap \mathcal{C}$ and general positive transformations

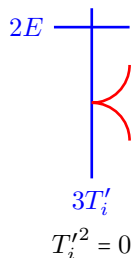
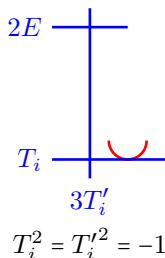
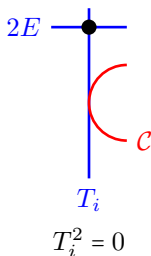


- ▶ Fiber: $2E + T'_i + \dots$
- ▶ Meridian: τ_i
- ▶ On π_1 : $x = y$



Nagata transformations

- ▶ Negative Nagata transformations at $T_i \cap E$ (a_i times) and general positive transformations

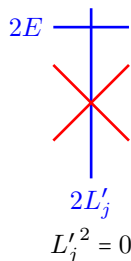
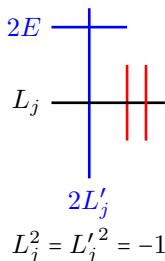
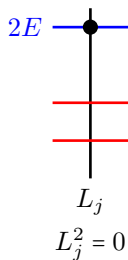


- ▶ Fiber: $2E + (2a_i + 1)T'_i + \dots$
- ▶ Meridian: $\tau_i e^{a_i}$
- ▶ On π_1 : $(xy)^{a_i} x = y(xy)^{a_i}$



Nagata transformations

- Negative Nagata transformations at $L_j \cap E$ (b_j times) and general positive transformations



- ▶ Fiber: $2E + 2b_j L'_j + \dots$
- ▶ Meridian: $\ell_j e^{b_j}$
- ▶ On π_1 : $(xy)^{b_j} = (yx)^{b_j}$

Main result

Theorem

There is a pencil with three special fibers

$$\mathcal{F}_1 : 2\mathcal{D} + \dots, \quad \mathcal{F}_2 : \sum (2a_i + 1)T'_i + 2 \sum b_j L'_j \dots, \quad \mathcal{F}_3 : \mathcal{C}_d + \sum q_j L_j$$

- *If there is an odd component in $\mathcal{F}_1$ then $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_d)$ is abelian.
Assume for now that there is no such component.*



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$$\langle x, y \mid (xy)^p x = y(xy)^p, [x^q, y] = 1, x^{rp} = (xy)^{(2p+1)(g+1+s)} \rangle$$



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Generators A, B, C, T, S , and relators:

- ▶ *T, S central*
- ▶ *$A^2 = ABC = T$*
- ▶ *$B^{2p+1} = T^p, C^q = S$*
- ▶ *$S^r = T^{g+1+s}$*



Conditions for existence

Theorem ($d \geq 4$)

$\exists \mathcal{C}_d$ hyperelliptic with a pencil of type $(2, 2p+1, q)$ if and only if
 $\exists s_1, r > 0$ such that

$$d = 2(2p+1)s_1 - rq, \text{ where } 0 < rq < d \iff d < 2(2p+1)s_1 < 2d$$

Note that if q is odd: $q \longleftrightarrow 2p+1$.



Conditions for existence

Corollary

1. For a fixed $\omega := (2, 2p + 1, q)$ and $d \gg 1$, $\exists \mathcal{C}_d$ hyperelliptic with an ω -pencil.



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3. For $d \geq 9$ and multiple of 3 $\exists \mathcal{C}_d$ hyperelliptic with a $(2, 3, 3)$ -pencil of type ; $\pi_1(\mathbb{P}^2 \setminus \mathcal{D}_d)$ is finite non abelian.
4. For $d \geq 8$ and even $\exists \mathcal{C}_d$ hyperelliptic with a $(2, 3, 4)$ -pencil; $\pi_1(\mathbb{P}^2 \setminus \mathcal{D}_d)$ is finite non abelian.
5. For $d \geq 6$ and $d \neq 8, 9, 10, 12, 15, 16, 23, 30$ $\exists \mathcal{C}_d$ hyperelliptic with an ω -pencil, for ω either $(2, 3, 5)$ or $(2, 5, 3)$; $\pi_1(\mathbb{P}^2 \setminus \mathcal{D}_d)$ is finite non abelian.



$(2, 3, 5)$ and $(2, 5, 3)$

For $d > 30$ we have always solutions.



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7	11	13	14	17	18	19	20
1·10-1·3	2·10-3·3		*	2·10-1·3	3·10-4·3		
2·6-1·5		3·6-1·5	*			4·6-1·5	5·6-2·5
21	22	24	25	26	27	28	29
*	*	3·10-2·3	4·10-5·3	*	3·10-1·3	*	5·10-7·3
*	*		5·6-1·5	*	7·6-3·5	*	9·6-5·5



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Corollary

For any $d \neq 1, 2, 3, 23$, there is a curve $\mathcal{C}$ of degree d such that $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ is finite non-abelian.



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Corollary

For any $d \neq 1, 2, 3, 23$, there is a curve $\mathcal{C}$ of degree d such that $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ is finite non-abelian.

Question

Is there a curve $\mathcal{C}$ of degree 23 such that $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ is finite non-abelian?

