Topology of complex plane curves: Zariski pairs

Enrique ARTAL BARTOLO

Departmento de Matemáticas
Universidad de Zaragoza

Geometry and Topology of Low Dimensional Manifolds
Burgo de Osma
September 2nd 2006

Contents

1. Introduction
2. The first Zariski pair
3. Arrangement of lines and arithmetic Zariski pairs
4. Arrangements of conics
5. Problems

Comparison

Three-dimensional Topology

Plane Complex Algebraic Geometry

Comparison

Three-dimensional Topology

- $\mathbb{S}^3$

Plane Complex Algebraic Geometry

Comparison

Three-dimensional Topology

- $\mathbb{S}^3$

Plane Complex Algebraic Geometry

- $\mathbb{P}^2$

Comparison

Three-dimensional Topology

- $\mathbb{S}^3$
- Knots and links

Plane Complex Algebraic Geometry

- $\mathbb{P}^2$

Comparison

Three-dimensional Topology

- $\mathbb{S}^3$
- Knots and links

Plane Complex Algebraic Geometry

- $\mathbb{P}^2$
- Projective plane curves

Comparison

Three-dimensional Topology

- $\mathbb{S}^3$
- Knots and links
- Any *3-manifold* is a ramified covering of $\mathbb{S}^3$ along a knot.

Plane Complex Algebraic Geometry

- $\mathbb{P}^2$
- Projective plane curves

Comparison

Three-dimensional Topology

- $\mathbb{S}^3$
- Knots and links
- Any *3-manifold* is a ramified covering of $\mathbb{S}^3$ along a knot.

Plane Complex Algebraic Geometry

- $\mathbb{P}^2$
- Projective plane curves
- Any *projective surface* is a ramified covering of $\mathbb{P}^3$ along a curve.

Comparison

Three-dimensional Topology

- $\mathbb{S}^3$
- Knots and links
- Any *3-manifold* is a ramified covering of $\mathbb{S}^3$ along a knot.
- Topological classification of knots and links

Plane Complex Algebraic Geometry

- $\mathbb{P}^2$
- Projective plane curves
- Any *projective surface* is a ramified covering of $\mathbb{P}^3$ along a curve.

Comparison

Three-dimensional Topology

- $\mathbb{S}^3$
- Knots and links
- Any *3-manifold* is a ramified covering of $\mathbb{S}^3$ along a knot.
- Topological classification of knots and links

Plane Complex Algebraic Geometry

- $\mathbb{P}^2$
- Projective plane curves
- Any *projective surface* is a ramified covering of $\mathbb{P}^3$ along a curve.
- Topological classification of reducible and irreducible curves

Topological properties

Three-dimensional topology appears directly:

- Algebraic links of singular points

Topological properties

Three-dimensional topology appears directly:

- Algebraic links of singular points
- Boundary of regular neighbourhoods

Topological properties

Three-dimensional topology appears directly:

- Algebraic links of singular points
- Boundary of regular neighbourhoods
- Using classical invariants: π_1, Alexander polynomials.

Necessary conditions

$\exists \psi : (\mathbb{P}^2, C_1) \xrightarrow{homeo} (\mathbb{P}^2, C_2)$?

- $\deg C_1 = \deg C_2$ (homology arguments).

Necessary conditions

$\exists\psi : (\mathbb{P}^2, C_1) \xrightarrow{homeo} (\mathbb{P}^2, C_2)$?

- $\deg C_1 = \deg C_2$ (homology arguments).
- $\#\operatorname{Sing}(C_1) = \#\operatorname{Sing}(C_2)$ (respecting topological types).

Necessary conditions

$\exists\psi : (\mathbb{P}^2, C_1) \xrightarrow{homeo} (\mathbb{P}^2, C_2)$?

- $\deg C_1 = \deg C_2$ (homology arguments).
- $\#\,\mathrm{Sing}(C_1) = \#\,\mathrm{Sing}(C_2)$ (respecting topological types).
- $C_i = \bigcup_{j=1}^{r} C_i^j$ and after relabelling $\deg C_1^j = \deg C_2^j$

Necessary conditions

$\exists\psi : (\mathbb{P}^2, C_1) \xrightarrow{homeo} (\mathbb{P}^2, C_2)$?

- $\deg C_1 = \deg C_2$ (homology arguments).
- $\#\,\mathrm{Sing}(C_1) = \#\,\mathrm{Sing}(C_2)$ (respecting topological types).
- $C_i = \bigcup_{j=1}^{r} C_i^j$ and after relabelling $\deg C_1^j = \deg C_2^j$
- Bijection between resolution graphs (respecting strict transforms of C_i^j).

Resolution and plumbing

$y^2 - x^3 = 0$

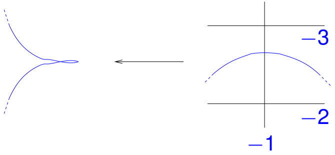


Figure: Resolution of the cusp

Resolution and plumbing

$y^2 - x^3 = 0$

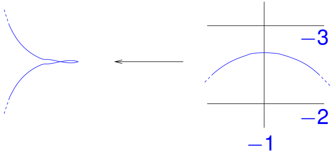


Figure: Resolution of the cusp

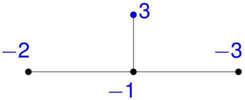


Figure: Dual graph of a cuspidal cubic

Resolution and plumbing

$y^2 - x^3 = 0$

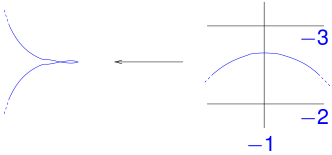


Figure: Resolution of the cusp

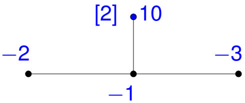


Figure: Dual graph of a cuspidal quartic

Resolution and plumbing

$y^2 - x^3 = 0$

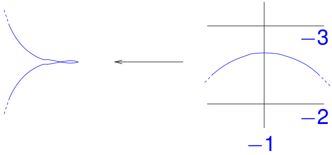


Figure: Resolution of the cusp

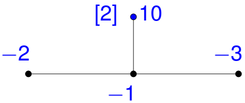


Figure: Dual graph of a cuspidal quartic

Union of tubular neighbourhoods $+$ plumbing $\Longrightarrow$ Waldhausen manifolds

Combinatorics

Definition

The combinatorics of a plane projective curve C is the dual graph of the embedded resolution with strict transforms marked.

Combinatorics

Definition

The combinatorics of a plane projective curve C is the dual graph of the embedded resolution with strict transforms marked.

Example

C irreducible: combinatorics $\equiv$ degree and topological types of singularities.

Combinatorics

Definition

The combinatorics of a plane projective curve C is the dual graph of the embedded resolution with strict transforms marked.

Example

C irreducible: combinatorics $\equiv$ degree and topological types of singularities.

Equivalent definition

The combinatorics of a plane projective curve C is the topological type of $(T(C), C)$, $T(C)$ regular neighbourhood.

Zariski pairs

- *C_1 and C_2 form a Zariski pair* if they have the same combinatorics but $(\mathbb{P}^2, C_1) \not\cong (\mathbb{P}^2, C_2)$.

Zariski pairs

- C_1 and C_2 form a *Zariski pair* if they have the same combinatorics but $(\mathbb{P}^2, C_1) \not\cong (\mathbb{P}^2, C_2)$.
- *Weak Zariski pair*: C_1, C_2 have same degree and topological types of singularities.

Zariski pairs

- C_1 and C_2 form a *Zariski pair* if they have the same combinatorics but $(\mathbb{P}^2, C_1) \not\cong (\mathbb{P}^2, C_2)$.
- *Weak Zariski pair*: C_1, C_2 have same degree and topological types of singularities.
- *Complement-type Zariski pair*: $\mathbb{P}^2 \setminus C_1 \not\cong \mathbb{P}^2 \setminus C_2$.

Zariski pairs

- C_1 and C_2 form a *Zariski pair* if they have the same combinatorics but $(\mathbb{P}^2, C_1) \not\cong (\mathbb{P}^2, C_2)$.
- *Weak Zariski pair*: C_1, C_2 have same degree and topological types of singularities.
- *Complement-type Zariski pair*: $\mathbb{P}^2 \setminus C_1 \not\cong \mathbb{P}^2 \setminus C_2$.
- π_1*-Zariski pair*: $\pi_1(\mathbb{P}^2 \setminus C_1) \not\cong \pi_1(\mathbb{P}^2 \setminus C_2)$.

Zariski pairs

- C_1 and C_2 form a *Zariski pair* if they have the same combinatorics but $(\mathbb{P}^2, C_1) \not\cong (\mathbb{P}^2, C_2)$.
- *Weak Zariski pair*: C_1, C_2 have same degree and topological types of singularities.
- *Complement-type Zariski pair*: $\mathbb{P}^2 \setminus C_1 \not\cong \mathbb{P}^2 \setminus C_2$.
- π_1*-Zariski pair*: $\pi_1(\mathbb{P}^2 \setminus C_1) \not\cong \pi_1(\mathbb{P}^2 \setminus C_2)$.
- *Alexander-type Zariski pair*: Alexander polynomials do not coincide.

Zariski pairs

- C_1 and C_2 form a *Zariski pair* if they have the same combinatorics but $(\mathbb{P}^2, C_1) \ncong (\mathbb{P}^2, C_2)$.
- *Weak Zariski pair*: C_1, C_2 have same degree and topological types of singularities.
- *Complement-type Zariski pair*: $\mathbb{P}^2 \setminus C_1 \ncong \mathbb{P}^2 \setminus C_2$.
- *π_1-Zariski pair*: $\pi_1(\mathbb{P}^2 \setminus C_1) \ncong \pi_1(\mathbb{P}^2 \setminus C_2)$.
- *Alexander-type Zariski pair*: Alexander polynomials do not coincide.
- *Libgober-type Zariski pair*: Characteristic varieties do not coincide.

Zariski pairs

- C_1 and C_2 form a *Zariski pair* if they have the same combinatorics but $(\mathbb{P}^2, C_1) \not\cong (\mathbb{P}^2, C_2)$.
- *Weak Zariski pair*: C_1, C_2 have same degree and topological types of singularities.
- *Complement-type Zariski pair*: $\mathbb{P}^2 \setminus C_1 \not\cong \mathbb{P}^2 \setminus C_2$.
- π_1*-Zariski pair*: $\pi_1(\mathbb{P}^2 \setminus C_1) \not\cong \pi_1(\mathbb{P}^2 \setminus C_2)$.
- *Alexander-type Zariski pair*: Alexander polynomials do not coincide.
- *Libgober-type Zariski pair*: Characteristic varieties do not coincide.
- *Arithmetic Zariski pair*: C_1, C_2 have Galois-conjugate equations in a number field [ACC05, Deg08].

[Zar29]

- Zariski-van Kampen method: Fig. .

[Zar29]

- Zariski-van Kampen method: Fig.. First application $\pi_1(\mathbb{P}^2 \setminus C) \cong \mathbb{Z}/d\mathbb{Z}$ if C smooth of degree d (for nodal curves it is a long history).

[Zar29]

- Zariski-van Kampen method: Fig.. First application $\pi_1(\mathbb{P}^2 \setminus C) \cong \mathbb{Z}/d\mathbb{Z}$ if C smooth of degree d (for nodal curves it is a long history).
- $X := \{t^3 - 3f_2(x,y,z)t - 2f_3(x,y,z) = 0\} \subset \mathbb{P}^3$.

[Zar29]

- Zariski-van Kampen method: ▸ Fig.. First application $\pi_1(\mathbb{P}^2 \setminus C) \cong \mathbb{Z}/d\mathbb{Z}$ if C smooth of degree d (for nodal curves it is a long history).
- $X := \{t^3 - 3f_2(x,y,z)t - 2f_3(x,y,z) = 0\} \subset \mathbb{P}^3$.
- $X \to \mathbb{P}^2, \quad [x:y:z:t] \mapsto [x:y:z]$ is a generic 3-covering ramified along $C := \{f_2^3 = f_3^2\} \subset \mathbb{P}^2$ ▸ Fig..

[Zar29]

- Zariski-van Kampen method: Fig. . First application $\pi_1(\mathbb{P}^2 \setminus C) \cong \mathbb{Z}/d\mathbb{Z}$ if C smooth of degree d (for nodal curves it is a long history).
- $X := \{t^3 - 3f_2(x,y,z)t - 2f_3(x,y,z) = 0\} \subset \mathbb{P}^3$.
- $X \to \mathbb{P}^2, \quad [x:y:z:t] \mapsto [x:y:z]$ is a generic 3-covering ramified along $C := \{f_2^3 = f_3^2\} \subset \mathbb{P}^2$ Fig. .
- C sextic with six cusps (on a conic!).

[Zar29]

- Zariski-van Kampen method: ▸ Fig.. First application $\pi_1(\mathbb{P}^2 \setminus C) \cong \mathbb{Z}/d\mathbb{Z}$ if C smooth of degree d (for nodal curves it is a long history).
- $X := \{t^3 - 3f_2(x,y,z)t - 2f_3(x,y,z) = 0\} \subset \mathbb{P}^3$.
- $X \to \mathbb{P}^2, \quad [x:y:z:t] \mapsto [x:y:z]$ is a generic 3-covering ramified along $C := \{f_2^3 = f_3^2\} \subset \mathbb{P}^2$ ▸ Fig..
- C sextic with six cusps (on a conic!).
- Space of curves of degree d is of dimension $\frac{d(d+3)}{2}$.

[Zar29]

- Zariski-van Kampen method: ▸ Fig.. First application $\pi_1(\mathbb{P}^2 \setminus C) \cong \mathbb{Z}/d\mathbb{Z}$ if C smooth of degree d (for nodal curves it is a long history).
- $X := \{t^3 - 3f_2(x,y,z)t - 2f_3(x,y,z) = 0\} \subset \mathbb{P}^3$.
- $X \to \mathbb{P}^2, \quad [x:y:z:t] \mapsto [x:y:z]$ is a generic 3-covering ramified along $C := \{f_2^3 = f_3^2\} \subset \mathbb{P}^2$ ▸ Fig..
- C sextic with six cusps (on a conic!).
- Space of curves of degree d is of dimension $\frac{d(d+3)}{2}$.
- Conditions imposed by a cusp: 4 (equation, derivatives and Hessian should vanish)

[Zar29]

- Zariski-van Kampen method: Fig.. First application $\pi_1(\mathbb{P}^2 \setminus C) \cong \mathbb{Z}/d\mathbb{Z}$ if C smooth of degree d (for nodal curves it is a long history).
- $X := \{t^3 - 3f_2(x,y,z)t - 2f_3(x,y,z) = 0\} \subset \mathbb{P}^3$.
- $X \to \mathbb{P}^2, \quad [x:y:z:t] \mapsto [x:y:z]$ is a generic 3-covering ramified along $C := \{f_2^3 = f_3^2\} \subset \mathbb{P}^2$ Fig..
- C sextic with six cusps (on a conic!).
- Space of curves of degree d is of dimension $\frac{d(d+3)}{2}$.
- Conditions imposed by a cusp: 4 (equation, derivatives and Hessian should vanish)-2 (the point can move on the plane) $= 2$.

[Zar29]

- Zariski-van Kampen method: ▸ Fig.. First application $\pi_1(\mathbb{P}^2 \setminus C) \cong \mathbb{Z}/d\mathbb{Z}$ if C smooth of degree d (for nodal curves it is a long history).
- $X := \{t^3 - 3f_2(x,y,z)t - 2f_3(x,y,z) = 0\} \subset \mathbb{P}^3$.
- $X \to \mathbb{P}^2, \quad [x:y:z:t] \mapsto [x:y:z]$ is a generic 3-covering ramified along $C := \{f_2^3 = f_3^2\} \subset \mathbb{P}^2$ ▸ Fig..
- C sextic with six cusps (on a conic!).
- Space of curves of degree d is of dimension $\frac{d(d+3)}{2}$.
- Conditions imposed by a cusp: 4 (equation, derivatives and Hessian should vanish)-2 (the point can move on the plane) $= 2$.
- Expected dimension of space of sextics with six cusps: $27 - 12 = 15$

[Zar29]

- Zariski-van Kampen method: ▸ Fig. . First application $\pi_1(\mathbb{P}^2 \setminus C) \cong \mathbb{Z}/d\mathbb{Z}$ if C smooth of degree d (for nodal curves it is a long history).
- $X := \{t^3 - 3f_2(x, y, z)t - 2f_3(x, y, z) = 0\} \subset \mathbb{P}^3$.
- $X \to \mathbb{P}^2, \quad [x : y : z : t] \mapsto [x : y : z]$ is a generic 3-covering ramified along $C := \{f_2^3 = f_3^2\} \subset \mathbb{P}^2$ ▸ Fig. .
- C sextic with six cusps (on a conic!).
- Space of curves of degree d is of dimension $\frac{d(d+3)}{2}$.
- Conditions imposed by a cusp: 4 (equation, derivatives and Hessian should vanish)-2 (the point can move on the plane) $= 2$.
- Expected dimension of space of sextics with six cusps: $27 - 12 = 15$
- Expected dimension of space of sextics $\{f_2^3 = f_3^2\}$: $5 + 9$

[Zar29]

- Zariski-van Kampen method: Fig.. First application $\pi_1(\mathbb{P}^2 \setminus C) \cong \mathbb{Z}/d\mathbb{Z}$ if C smooth of degree d (for nodal curves it is a long history).
- $X := \{t^3 - 3f_2(x,y,z)t - 2f_3(x,y,z) = 0\} \subset \mathbb{P}^3$.
- $X \to \mathbb{P}^2, \quad [x:y:z:t] \mapsto [x:y:z]$ is a generic 3-covering ramified along $C := \{f_2^3 = f_3^2\} \subset \mathbb{P}^2$ Fig..
- C sextic with six cusps (on a conic!).
- Space of curves of degree d is of dimension $\frac{d(d+3)}{2}$.
- Conditions imposed by a cusp: 4 (equation, derivatives and Hessian should vanish)-2 (the point can move on the plane) $= 2$.
- Expected dimension of space of sextics with six cusps: $27 - 12 = 15$
- Expected dimension of space of sextics $\{f_2^3 = f_3^2\}$: $5 + 9 + 1 = 15$.

- $\exists \pi_1(\mathbb{P}^2 \setminus C) \twoheadrightarrow \Sigma_3$

- $\exists \pi_1(\mathbb{P}^2 \setminus C) \twoheadrightarrow \Sigma_3$
- D sextic with six cusps and $\exists \pi_1(\mathbb{P}^2 \setminus D) \twoheadrightarrow \Sigma_3 \Longrightarrow$ the cusps are on a conic.

- $\exists \pi_1(\mathbb{P}^2 \setminus C) \twoheadrightarrow \Sigma_3$
- D sextic with six cusps and $\exists \pi_1(\mathbb{P}^2 \setminus D) \twoheadrightarrow \Sigma_3 \Longrightarrow$ the cusps are on a conic.
- [Zar37] Dual of a smooth conic: a sextic with nine cusps. Deformation argument: $\exists$ conic with six cusps not on a conic.

- $\exists \pi_1(\mathbb{P}^2 \setminus C) \twoheadrightarrow \Sigma_3$
- D sextic with six cusps and $\exists \pi_1(\mathbb{P}^2 \setminus D) \twoheadrightarrow \Sigma_3 \Longrightarrow$ the cusps are on a conic.
- [Zar37] Dual of a smooth conic: a sextic with nine cusps. Deformation argument: $\exists$ conic with six cusps not on a conic.
- Removing cusps: $aba = bab \rightsquigarrow a = b$.

- $\exists \pi_1(\mathbb{P}^2 \setminus C) \twoheadrightarrow \Sigma_3$
- D sextic with six cusps and $\exists \pi_1(\mathbb{P}^2 \setminus D) \twoheadrightarrow \Sigma_3 \Longrightarrow$ the cusps are on a conic.
- [Zar37] Dual of a smooth conic: a sextic with nine cusps. Deformation argument: $\exists$ conic with six cusps not on a conic.
- Removing cusps: $aba = bab \rightsquigarrow a = b$.
- Not in a conic: $\pi_1 \cong \mathbb{Z}/6\mathbb{Z}$.

- $\exists \pi_1(\mathbb{P}^2 \setminus C) \twoheadrightarrow \Sigma_3$
- D sextic with six cusps and $\exists \pi_1(\mathbb{P}^2 \setminus D) \twoheadrightarrow \Sigma_3 \Longrightarrow$ the cusps are on a conic.
- [Zar37] Dual of a smooth conic: a sextic with nine cusps. Deformation argument: $\exists$ conic with six cusps not on a conic.
- Removing cusps: $aba = bab \rightsquigarrow a = b$.
- Not in a conic: $\pi_1 \cong \mathbb{Z}/6\mathbb{Z}$.
- In a conic: $\pi_1 \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$.

- $\exists \pi_1(\mathbb{P}^2 \setminus C) \twoheadrightarrow \Sigma_3$
- D sextic with six cusps and $\exists \pi_1(\mathbb{P}^2 \setminus D) \twoheadrightarrow \Sigma_3 \implies$ the cusps are on a conic.
- [Zar37] Dual of a smooth conic: a sextic with nine cusps. Deformation argument: $\exists$ conic with six cusps not on a conic.
- Removing cusps: $aba = bab \rightsquigarrow a = b$.
- Not in a conic: $\pi_1 \cong \mathbb{Z}/6\mathbb{Z}$.
- In a conic: $\pi_1 \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$.
- [Zar31] Alexander polynomials: $t^2 - t + 1$ and 1 ([Lib82, Esn82, Art94])

$\pi_1(\mathbb{P}^2 \setminus C) \twoheadrightarrow \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$

$\mathbb{P}^2 \setminus C \xrightarrow{\phi} \mathbb{P}^1 \setminus \{[1:1]\} \equiv \mathbb{C}$
$\phi([x:y:z]) := [f_2^3 : f_3^2]$

$\pi_1(\mathbb{P}^2 \setminus C) \twoheadrightarrow \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$

$\mathbb{P}^2 \setminus C \xrightarrow{\phi} \mathbb{P}^1 \setminus \{[1:1]\} \equiv \mathbb{C}$
$\phi([x:y:z]) := [f_2^3 : f_3^2]$

$$\pi_1(\mathbb{P}^2 \setminus C) \twoheadrightarrow \pi_1(\mathbb{C})$$

$\pi_1(\mathbb{P}^2 \setminus C) \twoheadrightarrow \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$

$$\mathbb{P}^2 \setminus C \xrightarrow{\phi} \mathbb{P}^1 \setminus \{[1:1]\} \equiv \mathbb{C}$$
$$\phi([x:y:z]) := [f_2^3 : f_3^2]$$

$$\pi_1(\mathbb{P}^2 \setminus C) \twoheadrightarrow \pi_1(\mathbb{C})$$

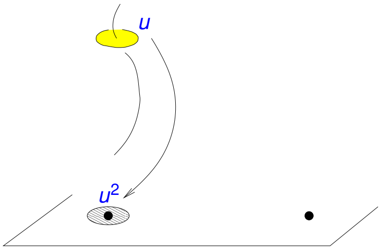


Figure: Orbifold group

$\pi_1(\mathbb{P}^2 \setminus C) \to \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$

$$\mathbb{P}^2 \setminus C \xrightarrow{\phi} \mathbb{P}^1 \setminus \{[1:1]\} \equiv \mathbb{C}$$
$$\phi([x:y:z]) := [f_2^3 : f_3^2]$$

$$\pi_1(\mathbb{P}^2 \setminus C) \twoheadrightarrow \pi_1(\mathbb{C})$$

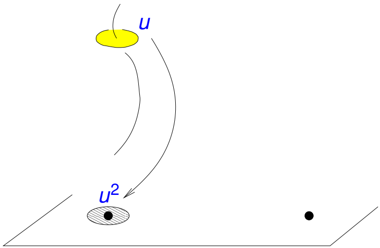


Figure: Orbifold group

$$\pi_1(\mathbb{P}^2 \setminus C) \twoheadrightarrow \pi_1(\mathbb{C}_{2,3})$$
$$\pi_1(\mathbb{C}_{2,3}) \equiv \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}.$$

Combinatorics of arrangement of lines $\equiv$ Intersection pattern

Combinatorics of arrangement of lines $\equiv$ Intersection pattern

Theorem (Rybnikov [Ryb98, ACCM05])

$\exists\ \mathscr{A}_1, \mathscr{A}_2$ with 13 lines and same combinatorics such that $\pi_1(\mathbb{P}^2 \setminus \mathscr{A}_1) \not\cong \pi_1(\mathbb{P}^2 \setminus \mathscr{A}_1)$ (not real equations).

Combinatorics of arrangement of lines $\equiv$ Intersection pattern

Theorem (Rybnikov [Ryb98, ACCM05])

$\exists$ $\mathscr{A}_1, \mathscr{A}_2$ with 13 lines and same combinatorics such that $\pi_1(\mathbb{P}^2 \setminus \mathscr{A}_1) \not\cong \pi_1(\mathbb{P}^2 \setminus \mathscr{A}_1)$ (not real equations).

Theorem ([ACCM03])

$\exists$ $\mathscr{A}_1, \mathscr{A}_2$ with 11 lines and same combinatorics which form an arithmetic Zariski pair (in $\mathbb{Q}(\sqrt{5})$).

Combinatorics of arrangement of lines $\equiv$ Intersection pattern

Theorem (Rybnikov [Ryb98, ACCM05])

$\exists\ \mathscr{A}_1, \mathscr{A}_2$ with 13 lines and same combinatorics such that $\pi_1(\mathbb{P}^2 \setminus \mathscr{A}_1) \not\cong \pi_1(\mathbb{P}^2 \setminus \mathscr{A}_1)$ (not real equations).

Theorem ([ACCM03])

$\exists\ \mathscr{A}_1, \mathscr{A}_2$ with 11 lines and same combinatorics which form an arithmetic Zariski pair (in $\mathbb{Q}(\sqrt{5})$).

Key point

[ACC03] Braid monodromy allows to recuperate topology

Arithmetic Zariski pair

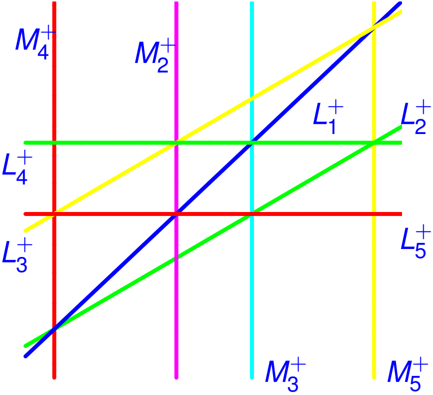


Figure: $\mathscr{A}^+$

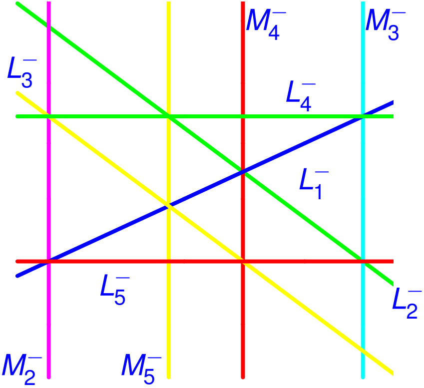


Figure: $\mathscr{A}^-$

Arithmetic Zariski pair

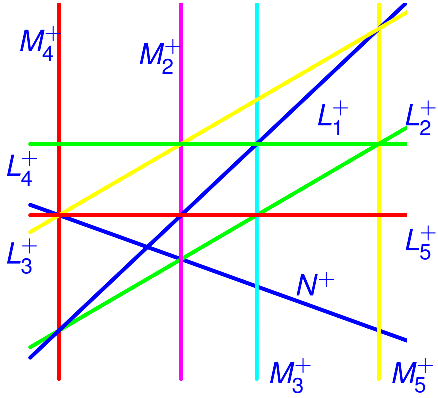


Figure: $\mathscr{B}^+$

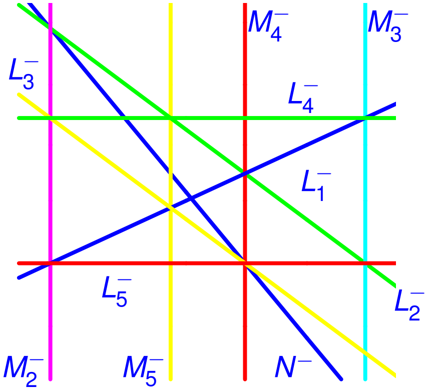


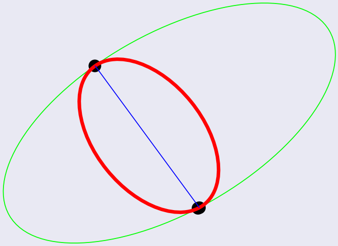
Figure: $\mathscr{B}^-$

Namba-Tsuchihashi example [NT04].

- Two conics are *bitangent* if they are smooth and are tangent at two different points.

Namba-Tsuchihashi example [NT04].

- Two conics are *bitangent* if they are smooth and are tangent at two different points.
- Take $C : \{c = 0\}$ a smooth conic, $\{\ell = 0\} =: L \pitchfork C$ line: C and $\{c + \alpha\ell^2 = 0\}$, $\alpha \in \mathbb{C}^*$, are bitangent.



- Combinatorics $\mathscr{C}_0$: C_1, C_2, C_3 conics such that $C_1 \pitchfork C_2$ and C_i, C_3 are bitangent, $i = 1, 2$.
- Moduli space of $\mathscr{C}_0$ is connected

- Combinatorics $\mathscr{C}_0$: C_1, C_2, C_3 conics such that $C_1 \pitchfork C_2$ and C_i, C_3 are bitangent, $i = 1, 2$.
- Moduli space of $\mathscr{C}_0$ is connected

Definition

C_1, C_2 smooth conics, $P \in \mathbb{P}^2$ special to C_1, C_2 (relative to C_P) if P is a double point of a conic C_P in the pencil $\Sigma(C_1, C_2)$.

Lemma

C_1, C_2, C_3 smooth conics. If P is special to C_i, C_3 (relative to $C_{P,i}$), $i = 1, 2$, then P is special to C_1, C_2 (relative to a product of lines in $\Sigma(C_{P,1}, C_{P,2})$)

Lemma

C_1, C_2, C_3 smooth conics. If P is special to C_i, C_3 (relative to $C_{P,i}$), $i = 1, 2$, then P is special to C_1, C_2 (relative to a product of lines in $\Sigma(C_{P,1}, C_{P,2})$)

Proof.

$$c_1 + c_3 = \ell_{13} m_{13}$$

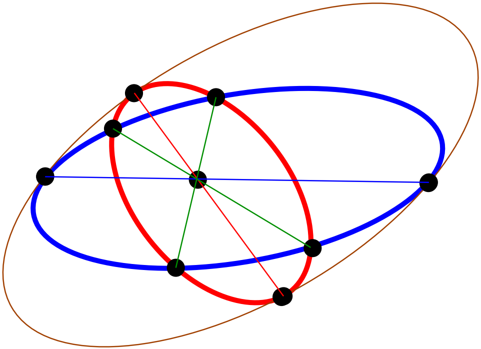
$$c_2 + c_3 = \ell_{23} m_{23}$$

$$c_1 - c_2 = \ell_{13} m_{13} - \ell_{23} m_{23} = \ell_{12} m_{12}$$

□

Consequence

If $\{C_1, C_2, C_3\}$ has combinatorics $\mathscr{C}_0$ then it has associated a point P special to C_1, C_2 (there are three such points)



Namba-Tsuchihashi combinatorics.

Combinatorics $\mathscr{C}_1$: C_1, C_2, C_3, C_4 conics such that:

Namba-Tsuchihashi combinatorics.

Combinatorics $\mathscr{C}_1$: C_1, C_2, C_3, C_4 conics such that:

- $C_3 \pitchfork C_4$

Namba-Tsuchihashi combinatorics.

Combinatorics $\mathscr{C}_1$: C_1, C_2, C_3, C_4 conics such that:

- $C_3 \pitchfork C_4$
- C_1, C_2, C_i in combinatorics $\mathscr{C}_0$, $i = 3, 4$.

Namba-Tsuchihashi combinatorics.

Combinatorics $\mathscr{C}_1$: C_1, C_2, C_3, C_4 conics such that:

- $C_3 \pitchfork C_4$
- C_1, C_2, C_i in combinatorics $\mathscr{C}_0$, $i = 3, 4$.
- P_i associated to C_1, C_2, C_i

Namba-Tsuchihashi combinatorics.

Combinatorics $\mathscr{C}_1$: C_1, C_2, C_3, C_4 conics such that:

- $C_3 \pitchfork C_4$
- C_1, C_2, C_i in combinatorics $\mathscr{C}_0$, $i = 3, 4$.
- P_i associated to C_1, C_2, C_i

Moduli space

Two connected components $\mathscr{M}_\pm$, depending on either $P_3 = P_4$ or $P_3 \neq P_4$.

Namba-Tsuchihashi combinatorics.

Combinatorics $\mathscr{C}_1$: C_1, C_2, C_3, C_4 conics such that:

- $C_3 \pitchfork C_4$
- C_1, C_2, C_i in combinatorics $\mathscr{C}_0$, $i = 3, 4$.
- P_i associated to C_1, C_2, C_i

Moduli space

Two connected components $\mathscr{M}_\pm$, depending on either $P_3 = P_4$ or $P_3 \neq P_4$.

Proposition ([ACT08])

C_1, C_2, C_3, C_4 is in $\mathscr{C}_1$
$P_3 = P_4 \iff$ there exists a conic C through the eight tacnodes

Idea of the proof

Construct two pencils of conics such that if C exists it is the intersection of the pencils.

Idea of the proof

Construct two pencils of conics such that if C exists it is the intersection of the pencils.

Consequences

Idea of the proof

Construct two pencils of conics such that if C exists it is the intersection of the pencils.

Consequences

- Existence of a tower of dihedral covers (if $P_3 = P_4$).

Idea of the proof

Construct two pencils of conics such that if C exists it is the intersection of the pencils.

Consequences

- Existence of a tower of dihedral covers (if $P_3 = P_4$).
- Alexander polynomials: $t^2 + 1$ and 1.

Idea of the proof

Construct two pencils of conics such that if C exists it is the intersection of the pencils.

Consequences

- Existence of a tower of dihedral covers (if $P_3 = P_4$).
- Alexander polynomials: $t^2 + 1$ and 1.
- Different number of positive dimensional irreducible components of characteristic varieties $\Rightarrow$ non-isomorphic fundamental groups (direct proof by Namba-Tsuchihashi).

- Understand topologically the influence of the position of singular points.

- Understand topologically the influence of the position of singular points.
- Study fundamental groups of the complements for arithmetic Zariski pairs.

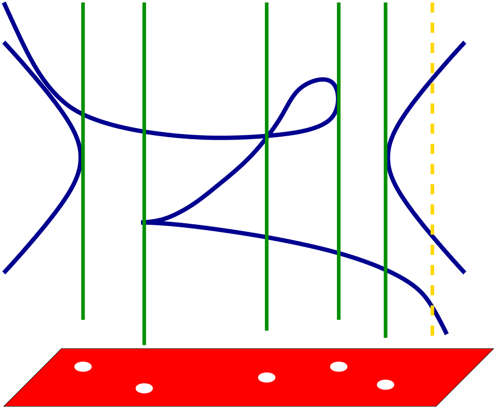
- Understand topologically the influence of the position of singular points.
- Study fundamental groups of the complements for arithmetic Zariski pairs.
- Example: There is an arithmetic Zariski pair such that the fundamental group of the complement is $\mathbb{Z} \times G$ (G Hurwitz group of the Klein quartic).

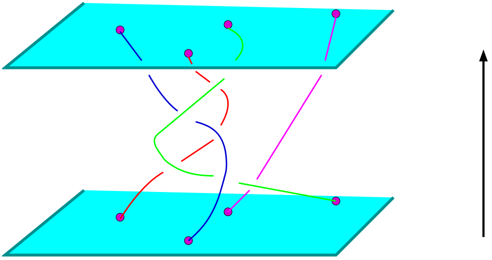
- E. Artal, J. Carmona, and J.I. Cogolludo, *Braid monodromy and topology of plane curves*, Duke Math. J. **118** (2003), no. 2, 261–278.
- ______, *Effective invariants of braid monodromy*, Trans. Amer. Math. Soc. (2005), aceptado.
- E. Artal, J. Carmona, J.I. Cogolludo, and M. Marco, *Topology and combinatorics of real line arrangements*, Compositio math. (2003), aceptado.
- ______, *Invariants of combinatorial line arrangements and Rybnikov's example*, Proceedings of 12th MSJ-IRI symposium "Singularity theory and its applications", 2005, accepted.
- E. Artal, J.I. Cogolludo, and H. Tokunaga, *A survey on Zariski pairs*, Algebraic geometry in East Asia—Hanoi 2005, Adv. Stud. Pure Math., vol. 50, Math. Soc. Japan, Tokyo, 2008, pp. 1–100.

- E. Artal, *Sur les couples de Zariski*, J. Algebraic Geom. **3** (1994), no. 2, 223–247.
- A. Degtyarev, *On deformations of singular plane sextics*, J. Algebraic Geom. **17** (2008), no. 1, 101–135.
- H. Esnault, *Fibre de Milnor d'un cône sur une courbe plane singulière*, Invent. Math. **68** (1982), no. 3, 477–496.
- A. Libgober, *Alexander polynomial of plane algebraic curves and cyclic multiple planes*, Duke Math. J. **49** (1982), no. 4, 833–851.
- M. Namba and H. Tsuchihashi, *On the fundamental groups of Galois covering spaces of the projective plane*, Geom. Dedicata **105** (2004), 85–105.
- G. Rybnikov, *On the fundamental group of the complement of a complex hyperplane arrangement*, Preprint available at `arXiv:math.AG/9805056`, 1998.

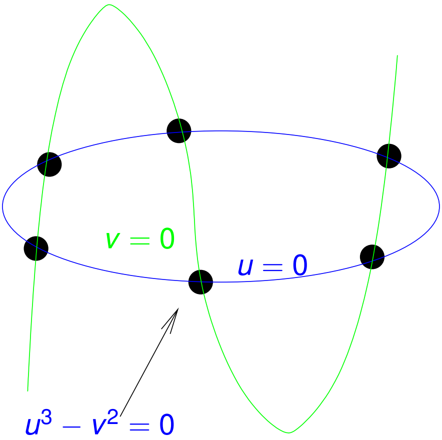
- O. Zariski, *On the problem of existence of algebraic functions of two variables possessing a given branch curve*, Amer. J. Math. **51** (1929), 305–328.
- ______, *On the irregularity of cyclic multiple planes*, Ann. Math. **32** (1931), 445–489.
- ______, *The topological discriminant group of a riemann surface of genus p*, Amer. J. Math. **59** (1937), 335–358.

$\pi_1(L \setminus C) \twoheadrightarrow \pi_1(\mathbb{P}^2 \setminus C)$





◂ Return



◂ Return