

A suspension formula for Denef-Loeser zeta functions

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Moduli, Hilbert Schemes and Singularities
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Settings

- ▶ $f(x_1, \dots, x_d) \in \mathbb{C}\{x_1, \dots, x_d\}$, $f(0) = 0$, $X = V(f)$.
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$$Z_{\text{top}}(f, \omega, s)_0 := \sum_{J \in \mathcal{P}_0} \chi(\mathring{E}_J) \prod_{i \in J} \frac{1}{\nu_i + N_i s}$$

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- ▶ A--CN-L-M: the topological zeta function is not topological.



Twisted zeta functions and suspensions

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- ▶ A--CN-L-M formula for $F(x_1, \dots, x_d, z) := z^2 + f(x_1, \dots, x_d)$:

$$\begin{pmatrix} \frac{s}{s+1} Z_{\text{top}}(F, \eta, s)_0 \\ Z_{\text{top}}^{(2)}(F, \eta, s)_0 \end{pmatrix} = \frac{1}{t} \begin{pmatrix} \frac{s}{s+1} \\ 1 \end{pmatrix} + \begin{pmatrix} 1 & -3 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} \frac{t}{t+1} Z_{\text{top}}(f, \omega, t)_0 \\ Z_{\text{top}}^{(2)}(f, \omega, t)_0 \end{pmatrix}$$

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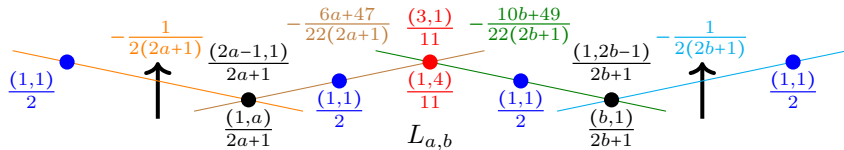
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- ▶ For $Z_{\text{top}}(f + z^2 + w^2, \eta \wedge dw, s)$, we need $Z_{\text{top}}^{(\ell)}(f, \omega, s)$, for $\ell = 1, 2$.



Du Bois-Michel singularities



$$N = 4a + 58$$

$$\nu = 4a + 15$$

$$N = 28$$

$$\nu = 7$$

$$N = 20$$

$$\nu = 5$$

$$N = 4b + 42$$

$$\nu = 4b + 11$$



Du Bois-Michel singularities

Theorem (Du Bois-Michel, 1991)

If $b \geq 5$, $b \neq a + 4$ then $L_{a,b}$ and $L_{b-4,a+4}$ are not equivalent and have the same Seifert form.

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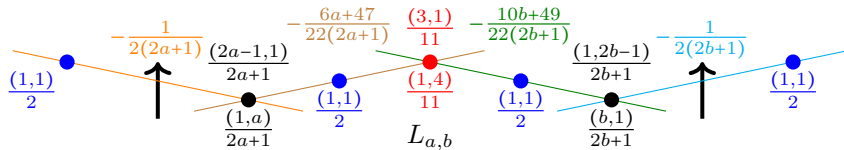
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Corollary

The topological zeta functions of the double suspensions of $L_{a,b}$ and $L_{b-4,a+4}$, $b \geq 5$, $b \neq a + 4$, do not coincide.



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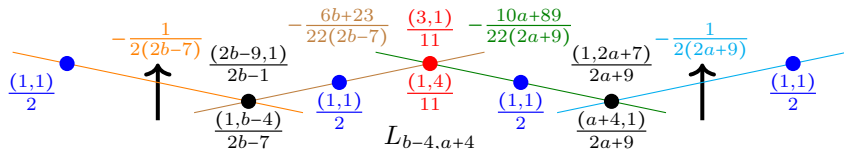
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$$\nu = 4b - 1$$

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$$\nu = 7$$

$$N = 20$$

$$\nu = 5$$

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$$\nu = 4a + 27$$



Plan

Loeser suggested formula for $Z_{\text{top}}(z^k - f, s)$

$$t = s + \frac{1}{k}$$

$$\frac{k-1}{k} \frac{s}{s+1} \frac{t+1}{t} \cdot Z_{\text{top}}(f, t)_0 - \frac{s}{s+1} \sum_{1 \neq \ell | k} \frac{(\ell+1)\phi(\ell)}{k} Z_{\text{top}}^{(\ell)}(f, t)_0 + \frac{1}{kt},$$



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Superisolated (SIS) and Lê-Yomdine singularities (LYS)

Key ingredients: Zeta functions of $z^p(z^k - f(x_1, \dots, x_d))$ with respect to some ω 's



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Goals

- ▶ Correct Loeser's formula and extend it to $Z_{\text{top}}^{(\ell)}$.



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- ▶ Correct Loeser's formula and extend it to $Z_{\text{top}}^{(\ell)}$.
- ▶ Compute the formula at motivic level (with holomorphic differentials) for $z^p(z^k - f(x_1, \dots, x_d))$
- ▶ Compute the zeta functions for SIS and LYS



Naive and motivic zeta functions

- ▶ X smooth, $E \subset X$ subvariety, $\mathcal{L}(X)_E := \{\varphi \text{ arc of } X \mid \varphi(0) \in E\}$.
- ▶ Grothendieck rings:

$$\mathbb{L} = [\mathbb{C}] \in K_0(\mathbf{Var}_{\mathbb{C}}) \rightarrow \mathcal{M}_{\mathbb{C}} = K_0(\mathbf{Var}_{\mathbb{C}})[\mathbb{L}^{-1}] \xrightarrow{\chi_{\text{top}}} \mathbb{Z}$$



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$$\mathbb{L} = [\mathbb{C}]^{\hat{\mu}} \in K_0^{\hat{\mu}}(\mathbf{Var}_{\mathbb{C}}) \rightarrow \mathcal{M}_{\mathbb{C}}^{\hat{\mu}} = K_0^{\hat{\mu}}(\mathbf{Var}_{\mathbb{C}})[\mathbb{L}^{-1}] \xrightarrow{\chi_{\text{top}}(\cdot, \alpha)} \mathbb{Z}$$

$\alpha : \hat{\mu} \rightarrow \mathbb{C}^*$ finite order.



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$$Z_{\text{mot}}(g, \omega, T)_{\mathbf{0}} := \sum_{J \in \mathcal{P}_0} (\mathbb{L} - 1)^{|J|-1} [\tilde{E}_J]^{\hat{\mu}} \prod_{i \in J} \frac{1}{\mathbb{L}^{\nu_i} T^{-N_i} - 1}$$

$$\tilde{E}_J \rightarrow \mathring{E}_J \text{ unramified } \gcd(N_i \mid l \in J)\text{-cyclic cover}$$


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- ▶ This proves that the topological zeta functions are well-defined:

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$$Z_{\text{top}}^{(\ell)}(g, \omega, s)_{\mathbf{0}} = \chi_{\text{top}}((\mathbb{L} - 1)Z_{\text{mot}}(g, \omega, \mathbb{L}^{-s})_{\mathbf{0}}, \alpha))$$

 α character of order ℓ .

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Change of variables and stratifications

Change of variables

X, Y smooth algebraic varieties, $\pi : Y \rightarrow X$ birational morphism,
 $F : X \rightarrow \mathbb{C}$ regular function, $p \in X$, ω maximal holomorphic form,
 $E := \pi^{-1}(p)$:

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Change of variables and stratifications

Change of variables

X, Y smooth algebraic varieties, $\pi : Y \rightarrow X$ birational morphism,
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Stratification Principle

- ▶ X smooth algebraic variety, $f : X \rightarrow \mathbb{C}$ regular function, $E \subset X$ subvariety.
- ▶ $E = \bigcup_{S \in \mathcal{S}} S$ finite prestratification,
- ▶ $\forall S \in \mathcal{S}, \forall y \in S, Z_{\text{naive}}(f, \omega, T; S) := Z_{\text{naive}}(f, \omega, T)_y$ does not depend on $y \in S$

$$Z_{\text{naive}}(f, \omega, T)_E = \sum_{S \in \mathcal{S}} [S] Z_{\text{naive}}(f, \omega, T; S).$$



Zeta functions of Lê-Yomdin singularities

Lê-Yomdin singularities

$V := F^{-1}(0)$, $F(\mathbf{x}) = F_m(\mathbf{x}) + F_{d+k}(\mathbf{x}) + \dots$, $\text{Sing } V_{\mathbb{P}}(F_m) \cap V_{\mathbb{P}}(F_{m+k}) = \emptyset$.

In particular, $C := V_{\mathbb{P}}(F_m)$ has isolated singularities.



Zeta functions of Lê-Yomdin singularities

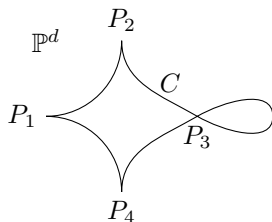
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Blow-up

$\pi : \hat{\mathbb{C}}^{d+1} \rightarrow \mathbb{C}^{d+1}$, chart $(x_1, \dots, x_d, z) \mapsto (x_1 z, \dots, x_d z, z)$

$$\begin{aligned}\pi \circ F &= z^m (F_m(x_1, \dots, x_d, 1) + z^k (F_{m+k}(x_1, \dots, x_d, 1) + \dots)) \\ &\equiv z^m (f(u_1, \dots, u_d) + z^k)\end{aligned}$$



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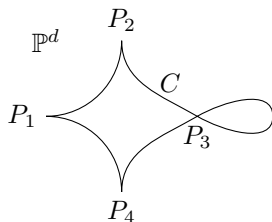
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$$\mathbb{P}^d \setminus C : z^m, z^d d\mathbf{x} \wedge dz$$

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$$P_i : z^m (z^k - f_p(\mathbf{u})), z^d d\mathbf{u} \wedge dz$$

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Motivic zeta function

$$\begin{aligned}Z_{\text{naive}}(F, d\mathbf{x}_{d+1}, T)_{\mathbf{0}} &= [\mathbb{P}^d \setminus C] Z_{\text{naive}}(z^m, z^d d\mathbf{x}_d \wedge dz, T)_{\mathbf{0}} \\ &\quad + [C \setminus \text{Sing } C] Z_{\text{naive}}(z^m u_1, z^d d\mathbf{u}_d \wedge dz, T)_{\mathbf{0}} \\ &\quad + \sum_{p \in \text{Sing } C} Z_{\text{naive}}(z^m (z^k + f_p), z^d d\mathbf{x}_d \wedge dz, T)_{\mathbf{0}}\end{aligned}$$



Suspensions

- ▶ $f(x_1, \dots, x_d), \omega_d,$
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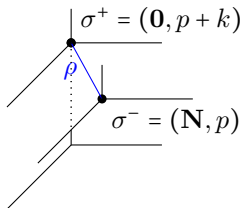
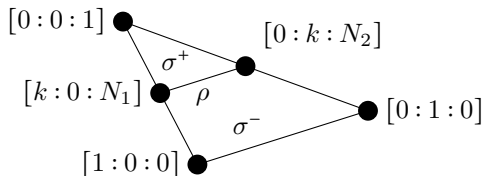
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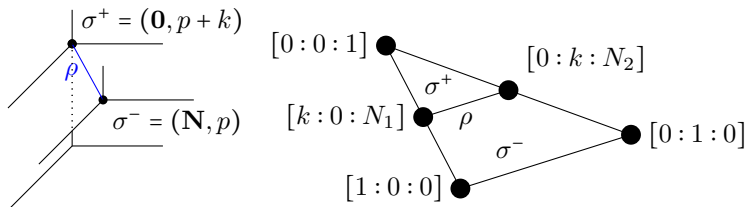
Arcs for suspensions of monomials

- $g_0 = z^k - \mathbf{x}^{\mathbf{N}}, g = z^p g_0, \omega_d = \mathbf{x}^{\nu-1} d\mathbf{x}, \omega_{d+1} = z^{\nu_z-1} \omega_d \wedge dz.$



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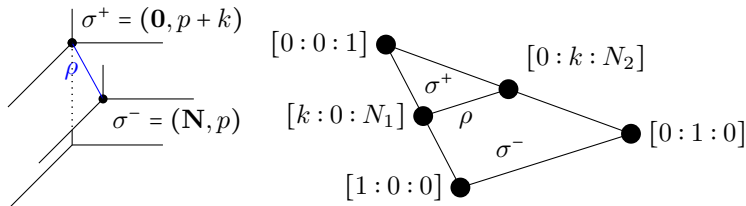


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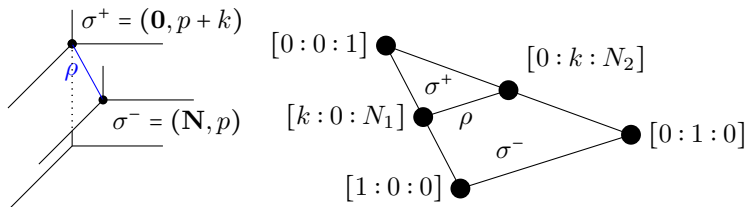


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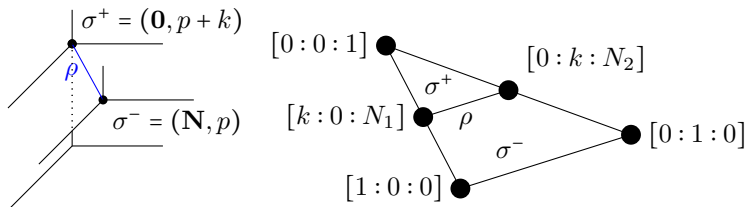


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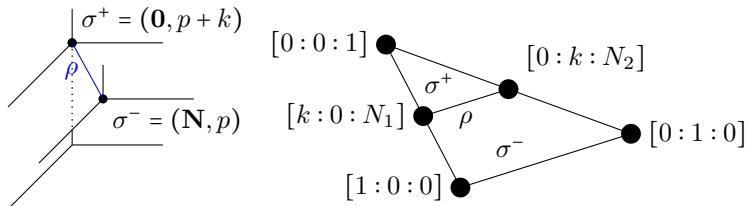
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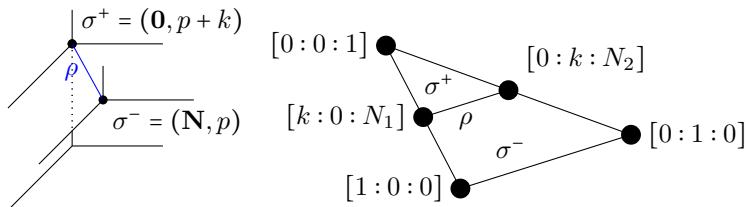


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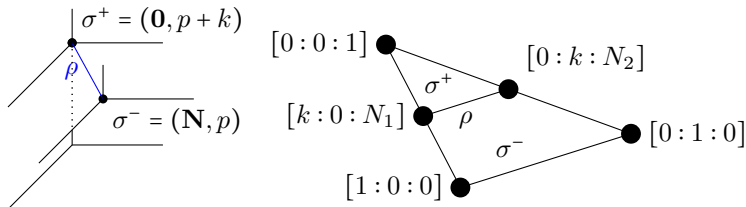


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- ▶ Compute measures and use generating functions.



Zeta functions for suspensions of monomials

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- ▶ $\sum_{d|n} J_k(d) = n^k$ (Jordan's totient functions)



Zeta functions for suspensions

- ▶ $\omega_d, g = z^k - f, 1 = k_1 < k_2 < \cdots < k_r = k$, divisors of k , $t = s + \frac{1}{k}$

$$\begin{pmatrix} \frac{s+1}{s} Z_{\text{top}}(F, \omega_d \wedge dz, s) \\ Z_{\text{top}}^{(k_2)}(F, \omega_d \wedge dz, s) \\ \vdots \\ Z_{\text{top}}^{(k_r)}(F, \omega_d \wedge dz, t) \end{pmatrix} =$$

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- ▶ $\ell \nmid k$, for some arithmetic values $m(k, \ell, 0)$:

$$Z_{\text{top}}^{(\ell)}(F, \omega_{d+1}, s)_{\mathbf{0}} = Z_{\text{top}}^{(\ell)}(f, \omega_d, t)_{\mathbf{0}} - \sum_{e|k} \frac{J_2(e)}{k} Z_{\text{top}}^{(\ell \text{lcm}(e, \ell_1 m(k, \ell, 0)))}(f, \omega_d, t)_{\mathbf{0}}.$$



Zeta functions for superisolated singularities ($k = 1$)

Motivic zeta function

$$\begin{aligned} Z_{\text{naive}}(F, d\mathbf{x}_{d+1}, T)_{\mathbf{0}} = & [\mathbb{P}^d \setminus C] Z_{\text{naive}}(z^m, z^d d\mathbf{x}_d \wedge dz, T)_{\mathbf{0}} \\ & + [C \setminus \text{Sing } C] Z_{\text{naive}}(z^m u_1, z^d d\mathbf{u}_d \wedge dz, T)_{\mathbf{0}} \\ & + \sum_{p \in \text{Sing } C} Z_{\text{naive}}(z^m(z^k + f_p), z^d d\mathbf{x}_d \wedge dz, T)_{\mathbf{0}} \end{aligned}$$



Zeta functions for superisolated singularities ($k = 1$)

Topological zeta function $k = 1$

$$\begin{aligned} Z_{\text{naive}}(F, d\mathbf{x}_{d+1}, T)_{\mathbf{0}} &= [\mathbb{P}^d \setminus C] Z_{\text{naive}}(z^m, z^d d\mathbf{x}_d \wedge dz, T)_{\mathbf{0}} \\ &\quad + [C \setminus \text{Sing } C] Z_{\text{naive}}(z^m u_1, z^d d\mathbf{u}_d \wedge dz, T)_{\mathbf{0}} \\ &\quad + \sum_{p \in \text{Sing } C} Z_{\text{naive}}(z^m(z^k + f_p), z^d d\mathbf{x}_d \wedge dz, T)_{\mathbf{0}} \end{aligned}$$

$$\begin{aligned} Z_{\text{top}}(F, d\mathbf{x}_{d+1}, s)_{\mathbf{0}} &= \frac{\chi_{\text{top}}(\mathbb{P}^d \setminus C)}{r} \\ &\quad + \frac{\chi_{\text{top}}(C \setminus \text{Sing } C)}{r(s+1)} \\ &\quad + \sum_{p \in \text{Sing } C} Z_{\text{top}}(z^m(z + f_p(\mathbf{u})), z^d d\mathbf{u}_d \wedge dz, s)_{\mathbf{0}} \end{aligned}$$

$$r = d + 1 + (m + 1)s$$



Zeta functions for superisolated singularities ($k = 1$)

Combine with $Z_{\text{top}}(z^m(z + f_p), z^d d\mathbf{u} \wedge dz, s)$

- ▶ Untwisted topological zeta function, $r = 1 + d + (1 + m)s$:

$$Z_{\text{top}}(F, d\mathbf{x}_{d+1}, s)_{\mathbf{0}} = \frac{d+1}{r-s} - \frac{s\chi(C)}{(r-s)(1+s)} \\ - \frac{s(r-s-1)}{r(r-s)(s+1)} \sum_{p \in \text{Sing } C} (1 + (r+1)Z_{\text{top}}(f_p, \omega_d, r)_{\mathbf{0}})$$



Zeta functions for superisolated singularities ($k = 1$)

Combine with $Z_{\text{top}}(z^m(z + f_p), z^d \mathbf{du} \wedge \mathbf{dz}, s)$

- ▶ Untwisted topological zeta function, $r = 1 + d + (1 + m)s$:

$$Z_{\text{top}}(F, \mathbf{dx}_{d+1}, s)_{\mathbf{0}} = \frac{d+1}{r-s} - \frac{s\chi(C)}{(r-s)(1+s)} \\ - \frac{s(r-s-1)}{r(r-s)(s+1)} \sum_{p \in \text{Sing } C} (1 + (r+1)Z_{\text{top}}(f_p, \omega_d, r)_{\mathbf{0}})$$

- ▶ $1 \neq \ell \mid m+1$:

$$Z_{\text{top}}^{(\ell)}(F, \mathbf{dx}_{d+1}, s)_{\mathbf{0}} = \sum_{p \in \text{Sing } C} \left(\frac{1}{r} + Z_{\text{top}}^{(\ell)}(f_p, \omega_d, r)_{\mathbf{0}} - \frac{r+1}{r} Z_{\text{top}}(f_p, \omega_d, p)_{\mathbf{0}} \right)$$



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- ▶ $1 \neq \ell \mid m$: $Z_{\text{top}}^{(\ell)}(F, \mathbf{dx}_{d+1}, s)_{\mathbf{0}} = \frac{s+1-r}{r-s} \sum_{p \in \text{Sing } C} Z_{\text{top}}^{(\ell)}(f_p, \omega_d, r)_{\mathbf{0}}$



Zeta functions for superisolated singularities ($k = 1$)

Combine with $Z_{\text{top}}(z^m(z + f_p), z^d \mathbf{du} \wedge \mathbf{dz}, s)$

- ▶ Untwisted topological zeta function, $r = 1 + d + (1 + m)s$:

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- ▶ $1 \neq \ell \mid m+1$:

$$Z_{\text{top}}^{(\ell)}(F, \mathbf{dx}_{d+1}, s)_{\mathbf{0}} = \sum_{p \in \text{Sing } C} \left(\frac{1}{r} + Z_{\text{top}}^{(\ell)}(f_p, \omega_d, r)_{\mathbf{0}} - \frac{r+1}{r} Z_{\text{top}}(f_p, \omega_d, p)_{\mathbf{0}} \right)$$

- ▶ $1 \neq \ell \mid m$: $Z_{\text{top}}^{(\ell)}(F, \mathbf{dx}_{d+1}, s)_{\mathbf{0}} = \frac{s+1-r}{r-s} \sum_{p \in \text{Sing } C} Z_{\text{top}}^{(\ell)}(f_p, \omega_d, r)_{\mathbf{0}}$
- ▶ $\ell \nmid m, m+1$:

$$Z_{\text{top}}^{(\ell)}(F, \mathbf{dx}_{d+1}, s)_{\mathbf{0}} = - \sum_{p \in \text{Sing } C} Z_{\text{top}}^{(\frac{\ell}{\gcd(\ell, m+1)}}(f_p, \omega_d, r)_{\mathbf{0}}$$

