Configuraciones de rectas: topología, grupo fundamental y combinatoria

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XII Encuentro de Topología
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Topología de rectas: segunda parte.

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Definitions: arrangements, combinatorics

Classical Zariski pairs for line arrangements

Graph manifolds in Algebraic Geometry and linking number

Guerville-Viu combinatorics types

Last developments

Section 1

Definitions: arrangements, combinatorics

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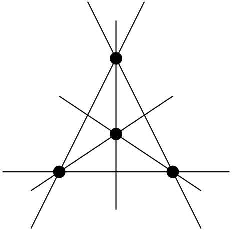

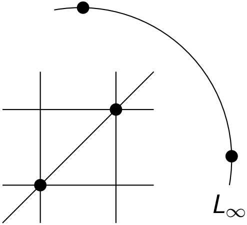

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Homological invariants

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$$H_1(\mathscr{C}) := \frac{\bigoplus_{\ell\in\mathscr{L}} \mathbb{Z}x_\ell}{\left\langle \sum_{\ell\in\mathscr{L}} x_\ell \right\rangle}, \qquad b_1(M(\mathscr{A})) = \#\mathscr{A} - 1.$$

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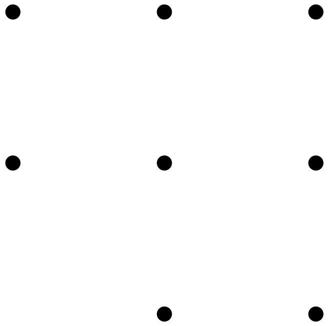
Theorem (Orlik-Solomon)

The ring structure of $H^*(M(\mathscr{A});\mathbb{Z})$ depends only on $\mathscr{C}(\mathscr{A})$.

Section 2

Classical Zariski pairs for line arrangements

McLane and Rybnikov combinatorics



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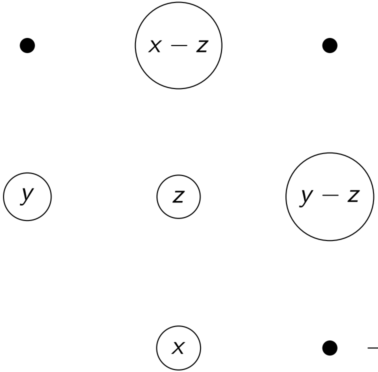

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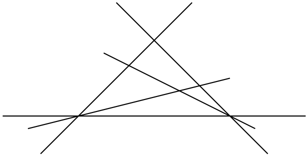

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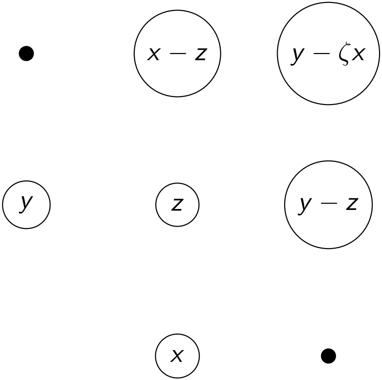

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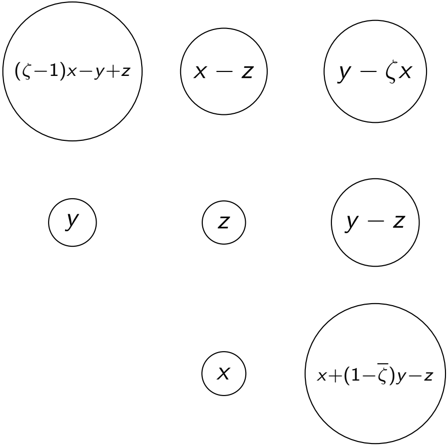

$\zeta^2+\zeta+1=0$

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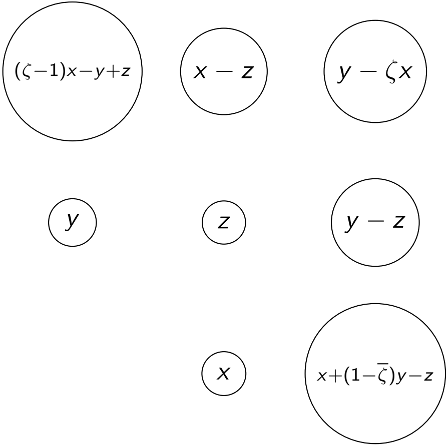

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$$(y^3-z^3)(z^3-x^3)(x^2+xy+y^2)=0$$

McLane and Rybnikov combinatorics

Theorem (Rybnikov)

$\nexists \Phi : \pi_1(M(\mathscr{N}_+)) \xrightarrow{\cong} \pi_1(M(\mathscr{N}_-))$ *inducing the identity on* $H_1(\mathscr{N};\mathbb{Z})$.

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$\mathscr{R}$ union of two ordered copies of $\mathcal{N}$ with three concurrent lines in common and generic intersections for the other lines: $\mathscr{A}_{++}$ and $\mathscr{A}_{+-}$.

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Theorem (Rybnikov (1994/2011),_-Carmona-Cogolludo-Marco(2006))

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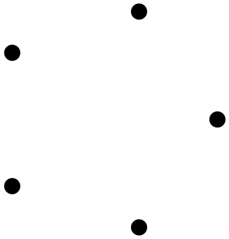

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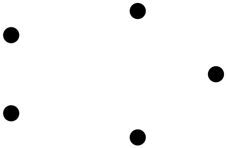
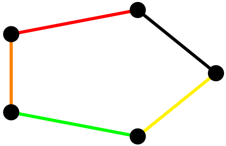


$\mathbb{Q}(\sqrt{5})$

Pentagon



BROGY

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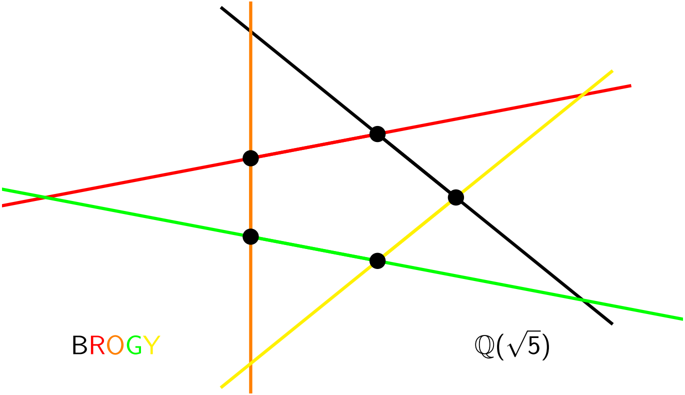

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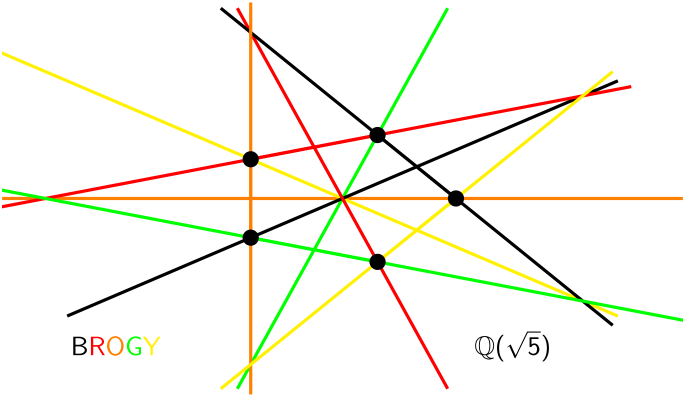
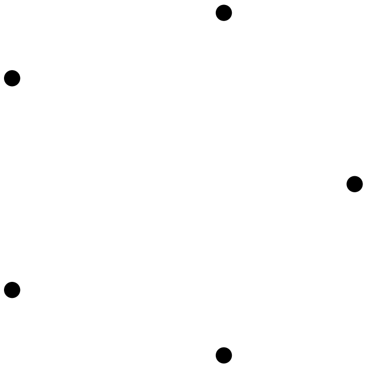


Pentagram



$$\sqrt{5} \mapsto -\sqrt{5}$$

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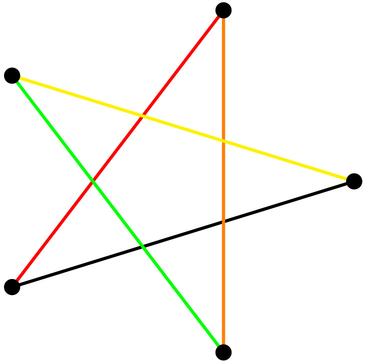

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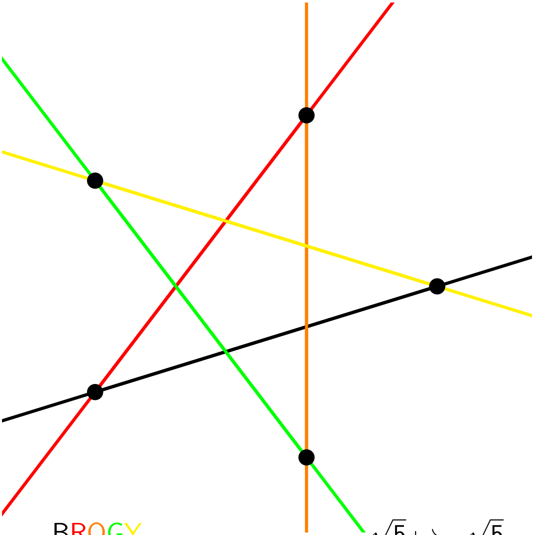

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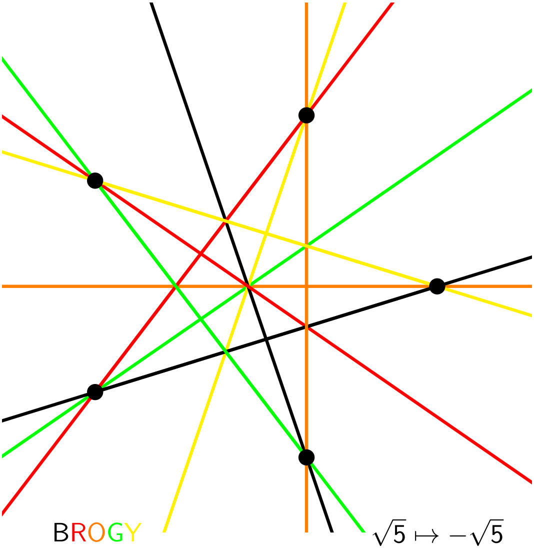
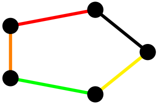
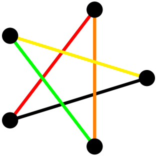


Pentagon and pentagram

BROGY



$\mathscr{Z}_+$



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BROGY

$\mathscr{L}_+$ $\mathscr{L}_-$

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Remarks

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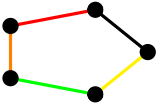
BROGY



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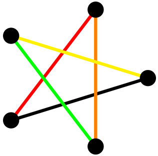

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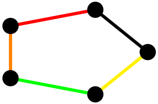

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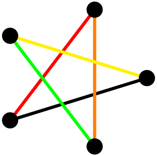

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Section 3

Graph manifolds in Algebraic Geometry and linking number

Graph manifolds

Tubular neighborhoods

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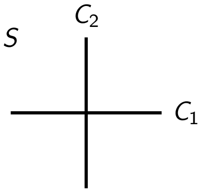
- $\partial T(C) \to C$ $\mathbb{S}^1$-fiber bundle determined by g (genus of the curve) and $e := C \cdot C$ (Euler number).

Graph manifolds

Regular neighborhoods

- S complex surface, $D \subset S$ normal crossing divisor: smooth irreducible components, only normal crossings ($xy = 0$).



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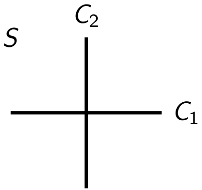

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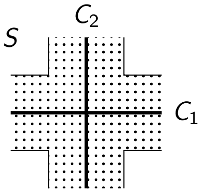

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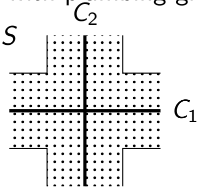
- $\partial T(D)$ graph manifold with plumbing graph Γ.



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- Edges of Γ:(P, L) $(P \in L,\ P \in \mathscr{P}_{>2})$ and $\{L_1, L_2\}$ $(L_1 \cap L_2 \in \mathscr{P}_{=2})$.

Graph manifold of a line arrangement

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- $\sigma : S \to \mathbb{P}^2$ blowing-up of $\mathscr{P}_{>2}$: $M(\mathscr{A}) = S \setminus \sigma^{-1}(\bigcup \mathscr{A})$.
- $\sigma^{-1}\left(\bigcup \mathscr{A}\right) = \underbrace{\bigcup_{L \in \mathscr{A}} \tilde{L}}_{\text{strict transforms}} \cup \underbrace{\bigcup_{P \in \mathscr{P}_{>2}} E_P}_{\text{exceptional components}}.$
- Vertices of Γ: E_P $(g = 0, e = -1)$, $\tilde{L}$ $(g = 0, e = 1 - \#\mathscr{P}_{>2} \cap L)$.
- Edges of Γ:(P, L) $(P \in L,\ P \in \mathscr{P}_{>2})$ and $\{L_1, L_2\}$ $(L_1 \cap L_2 \in \mathscr{P}_{=2})$.
- $\partial T(\mathscr{A})$ determines and is determined by the combinatorics of $\mathscr{A}$ (Waldhausen, Neumann, di Pasquale).

Coverings and characters

Lemma

S smooth compact surface, D NCD, $\check{\pi} : \check{T} \to S \setminus D$ finite covering extends to a finite branched covering $\pi : T \to S$ (T may be singular) with branching locus contained in D.

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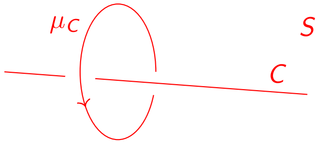
Lemma

C rational irreducible component of D: $\pi^{-1}(C)$ has n irreducible components $\iff \xi(\mu_C)=1$ and $\xi(\mu_{C'})=1$, $\forall C'\cap C\neq\emptyset$.



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Theorem (_ (2014), _-Florens-Guerville (2017))

$\mathscr{A}$ arrangement such that $\mathscr{C}(\mathscr{A}) = \mathscr{C}$, $(\mathscr{C}, \xi, \gamma)$ inner cyclic invariant. There is a way to push γ into a cycle $\tilde{\gamma} \in H_1(M(\mathscr{A}); \mathbb{Z})$ such that $\mathcal{I}(\mathscr{A}, \xi, \gamma) := \xi(\tilde{\gamma})$ does not depend on $\tilde{\gamma}$ and is an invariant of the ordered oriented topology of $\mathscr{A}$.

Inner cyclic invariant II

Remarks

- Suppose $\ell_1, \ell_2, \ell_3 \in \mathscr{L}$ are in a cycle corresponding to $L_1, L_2, L_3 \in \mathscr{A}$, ξ character of order 2.

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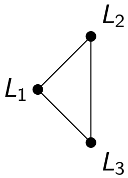

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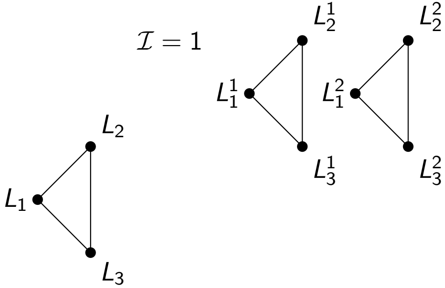

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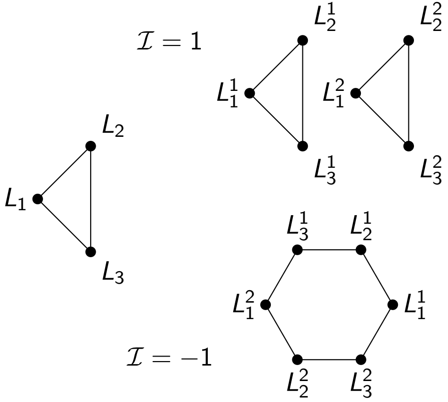

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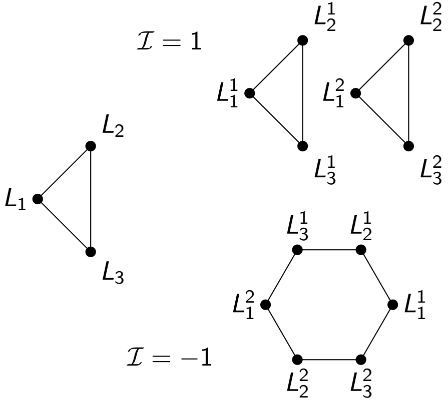
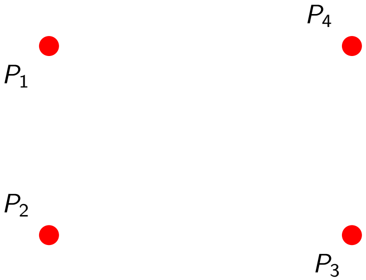


- It is in fact an invariant of the topology of $M(\mathscr{A})$.

Guerville combinatorics



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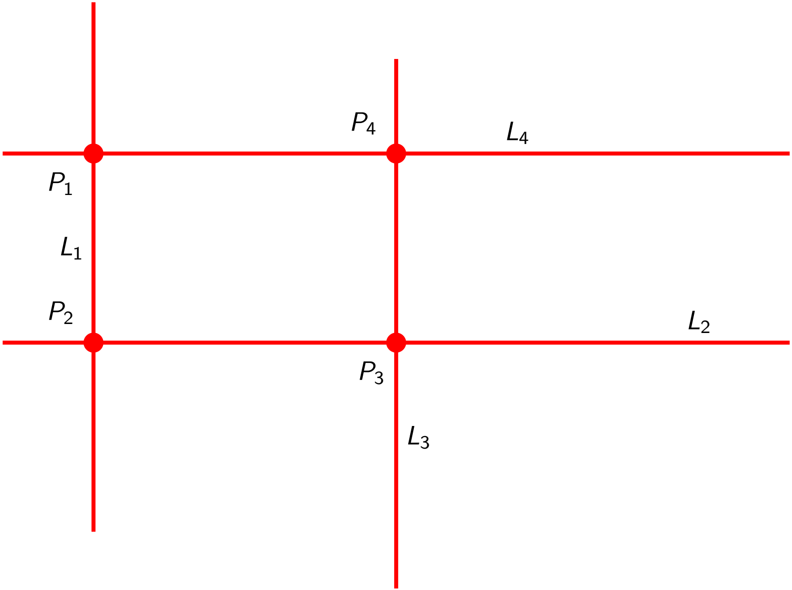

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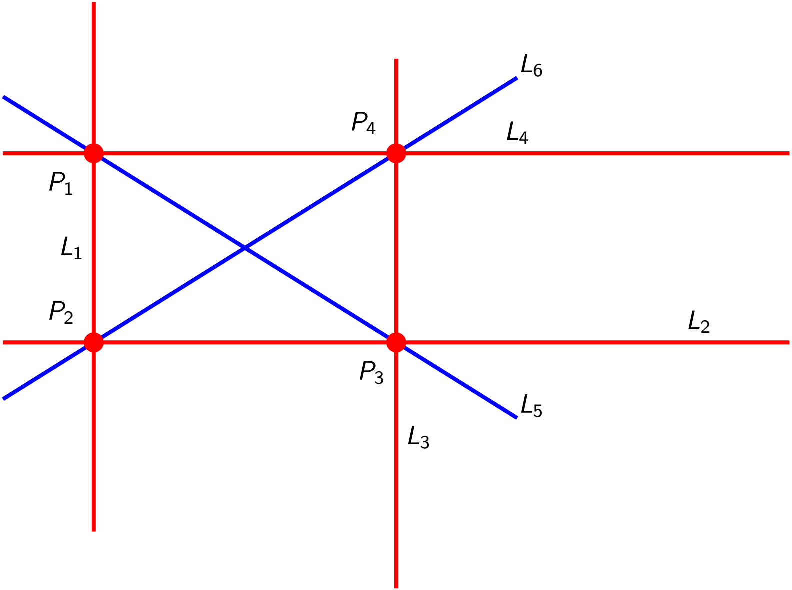

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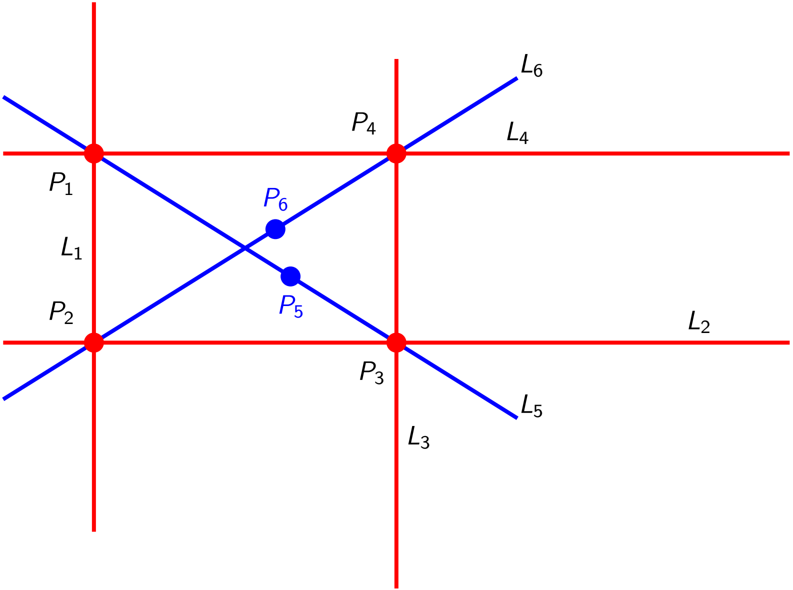

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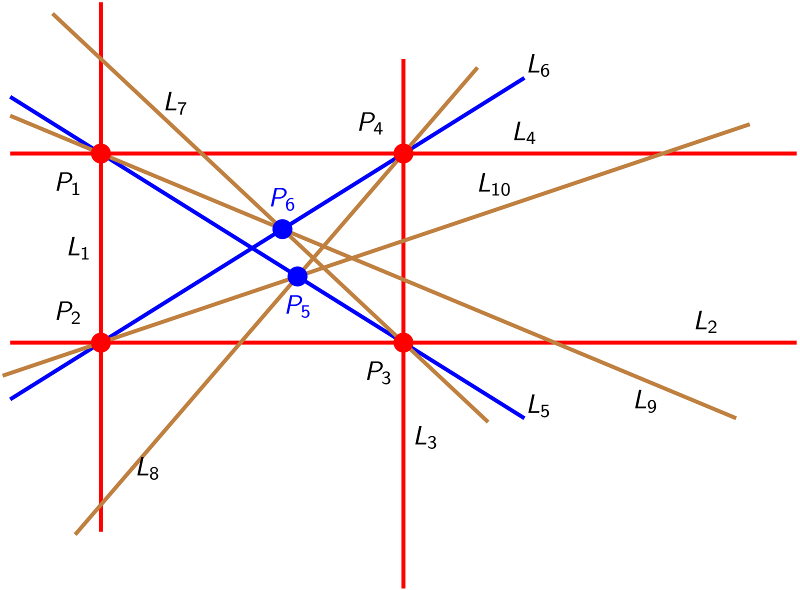

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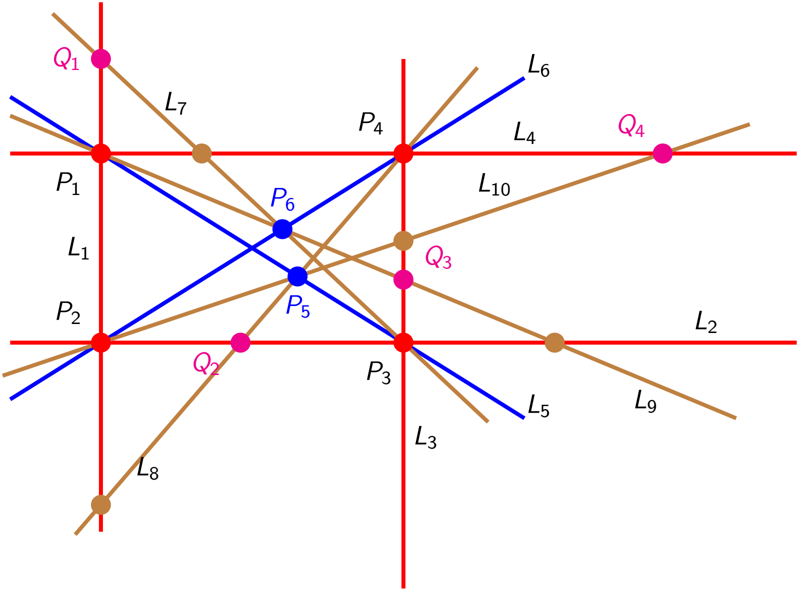

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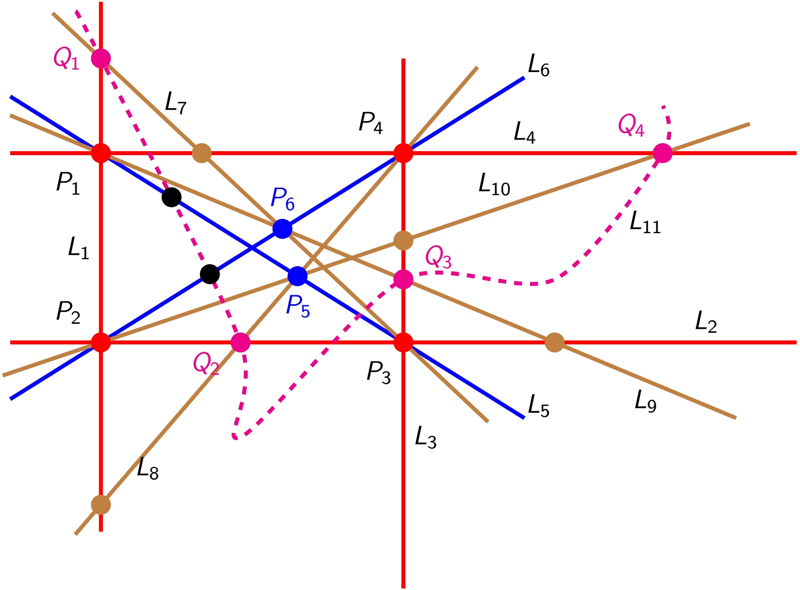

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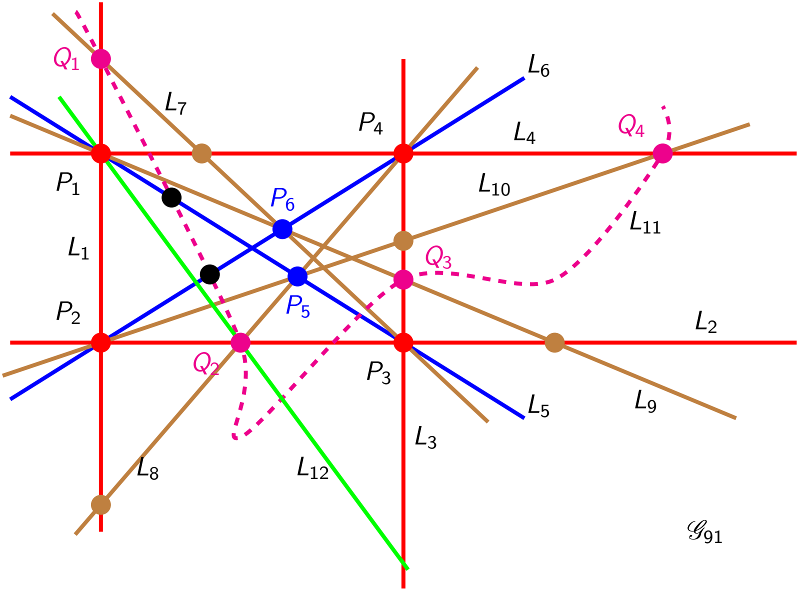

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Theorem (_-Cogolludo-Guerville-Marco (2017))

$\pi_1(M(\mathscr{A}_{\zeta_5})) \not\cong \pi_1(M(\mathscr{A}_{\zeta_5^2}))$ *(first example of non-isomorphic fundamental groups of complement of curves having the same profinite completion).*

Section 4

Guerville-Viu combinatorics types

Definitions

Definition and inner cyclic invariant

A combinatorics $\mathscr{C}$ is is of Guerville-Viu type if

- There are ℓ_1, ℓ_2, ℓ_3 such that $\ell_i \cap \ell_j$ is a double point.
- Any multiple point outside ℓ_j is double
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In particular such a combinatorics has an odd number of lines.

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Then $(\mathscr{C}, \xi, \gamma)$ defines an inner cyclic invariant.

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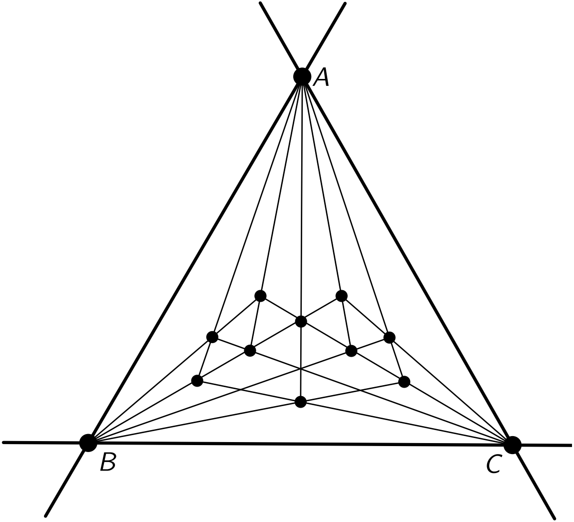
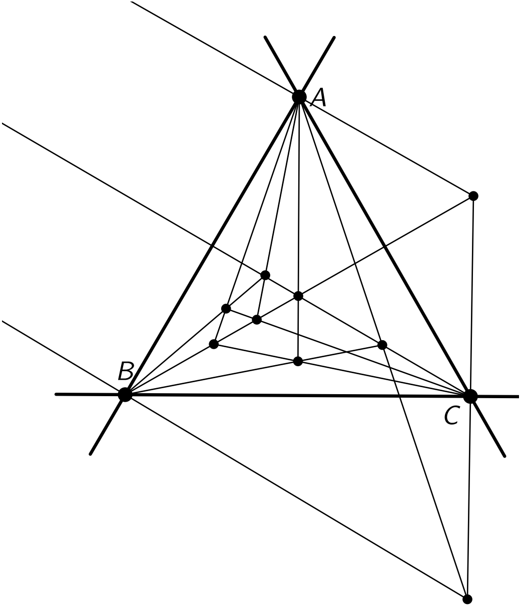
Theorem (Guerville-Viu)

Let $\mathscr{A}$ be a realization with real equations. Let T be the cell-decomposition of $\check{\mathbb{RP}}^2$ determined by L_1, L_2, L_3. The parity ε of the number of dual points $L \neq L_j$ in each cell is constant and $\mathcal{I}(\mathscr{A}, \xi, \gamma) = \varepsilon$.

Dual point arrangement of $\mathscr{V}_1$



Dual point arrangement of $\mathscr{V}_2$



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Theorem (_-Guerville-Viu)

The torsion of the graded Lie algebra associated to the lower central series of $\pi_1(M(\mathscr{V}_1))$ and $\pi_1(M(\mathscr{V}_2))$ is different. In particular, these groups are not isomorphic.

Section 5

Last developments

Zariski pairs with the same fundamental group

Theorem (Oka-Sakamoto (1978))

$C_1, C_2 \subset \mathbb{C}^2$ *affine curves,* $\# C_1 \cap C_2 = \deg C_1 \cdot \deg C_2$*. Then*
$\pi_1(\mathbb{C}^2 \setminus (C_1 \cup C_2)) \cong \pi_1(\mathbb{C}^2 \setminus C_1) \times \pi_1(\mathbb{C}^2 \setminus C_2)$

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- $(\mathscr{C}, \xi, \gamma)$ inner cyclic invariant, ξ of order *n*, $\mathscr{A}$ realization of $\mathscr{C}$ in a number field $\mathbb{K} \subset \mathbb{C}$ containing the *n*-roots of unity.

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- It is possible to compute $\mathcal{I}(\mathscr{A}, \xi, \gamma)$ algebraically such that if $\sigma \in \mathrm{Gal}(\mathbb{K}/\mathbb{Q})$ then $\mathcal{I}(\sigma(\mathscr{A}), \xi, \gamma) = \sigma(\mathcal{I}(\mathscr{A}, \xi, \gamma))$.

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- The inner cyclic invariant determines an element $\xi \otimes \gamma \in H^*(\mathscr{C}; \mathbb{C}^*) \otimes H_1(\Gamma; \mathbb{Z})$.

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- It can explain the oriented ordered topological difference for McLane arrangements.

¡¡Gracias por su atención!!

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