Arithmetic Zariski pairs with possibly non-isomorphic fundamental groups

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Topology and Combinatorics

Let $C \subset \mathbb{P}^2 =: \mathbb{P}^2(\mathbb{C})$ be a plane projective curve.

- **Topological type of C:** Homeomorphism type of $(\mathbb{P}^2, C)$
- **Combinatorial type of C:** Homeomorphism type of $(R(C), C)$, $R(C)$ a regular neighbourhood of C. Completely determined by a weighted graph.

Zariski pairs

Zariski pair: Two curves having the same combinatorics but different topological types.
Arithmetic Zariski pair: A Zariski pair (C_1, C_2) such that

- $C_i = \{f_i(x, y, z) = 0\}$
- $f_i(x, y, z) \in \mathbb{K}[x, y, z]$, $\mathbb{K}$ a number field
- $\exists \sigma \in \mathrm{Gal}(K/\mathbb{Q})$ such that $f_2 = f_1^\sigma$.

Galois-conjugate non-homeomorphic algebraic varieties exist [5, 1] and so do arithmetic Zariski pairs [3, 4].

Invariants

The following invariants are widely used to prove that two curves form a Zariski pair

- $\pi_1(\mathbb{P}^2 \setminus C)$
- **Alexander polynomial of C**
- **Characteristic varieties of C**

They are not valid for arithmetic Zariski pairs since $\pi_1(\mathbb{P}^2 \setminus C_i)$ have isomorphic profinite completions

Braid monodromy

- $P := [0 : 1 : 0] \in \mathbb{P}^2$, $P \in L := \{z = 0\}$
- The tangent cone of C at P is contained in L.
- $f(x, y) := y^n + \sum_{j=1}^n f_j(x) y^{n-j} = 0$, reduced normalized affine equation of C.
- $D := \{t \in \mathbb{C} \mid f(t, y) = 0 \text{ has } < n \text{ solutions}\}$, $\#D := r$.

If $t \in \mathbb{C} \setminus D$ then $f(t, y) = 0$ has exactly n roots and a natural morphism $\nabla : \pi_1(\mathbb{C} \setminus D; t_0) \to \mathbb{B}_n$ is defined: the braid monodromy.

- $\nabla \leftrightarrow \tau \in (\mathbb{B}_n)^r$
- $\tau_1, \tau_2 \in (\mathbb{B}_n)^r$ represent the same braid monodromy if and only if they are in the same orbit by the action of $\mathbb{B}_n \times \mathbb{B}_r$:
  - $\mathbb{B}_n$ acts by simultaneous conjugation.
  - $\mathbb{B}_r$ acts by Hurwitz moves.

Main Tool

Definition. The fibered curve C^φ for a braid monodromy is a triple formed by $(C, L, \{L_t\}_{t \in D})$ where $L_t := \{x = tz\}$

Theorem ([2]). *Let C_1, C_2 be two curves with fibered curves C_i^φ for some braid monodromies. Suppose there exists an oriented homeomorphism $\Phi : \mathbb{P}^2 \to \mathbb{P}^2$ such that $\Phi(C_1^\varphi) = C_2^\varphi$ (preserving orientations). Then C_1 and C_2 have the same braid monodromy.*

Strategy. Given two combinatorially equivalent curves with different braid monodromies, their fibered curves form a Zariski pair.

How can we prove that two elements in $(\mathbb{B}_n)^r$ are not in the same orbit?
Work with a finite representation $\mathbb{B}_n \to G$ and the action of $G \times \mathbb{B}_r$ on G^r (with finite orbits!).

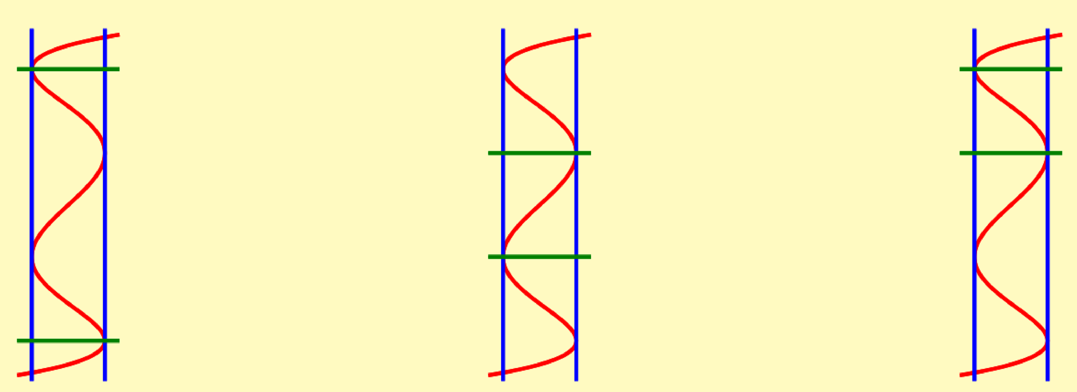
Example. Let $\rho : \mathbb{B}_n \to \mathrm{GL}(n-1, \mathbb{Z}[t^{\pm}1])$ the reduced Burau representation. Replace $\mathbb{Z}[t^{\pm}1]$ by either $\mathbb{Z}/m$ or $\mathbb{F}_{p^k}$ (specializing t to a unit).

Explicit examples

First pair $Z_{1,\pm}$: Curves consisting of three concurrent lines (L_∞, L_1, L_2), a quintic C_5 with two singular points Q, R and a smooth conic C_2: (C_5, Q) of type $\mathbb{D}_6$ and (C_5, R) of type $\mathbb{A}_4$, the tangent branches of Q are tangent to L_1, $(C_5 \cdot L_\infty)_R = 5$. The line L_2 is bitangent to C_5 and C_2 is tangent to C at Q, R and passes through one of the bitangencies.

Second pair $Z_{2,\pm}$: Curves consisting of three concurrent lines: L_∞, L_1, L_2 a quintic C_5 with one singular point Q, M a line: (C_5, Q) has local equation $u^4 = v^5$ and is tangent to L_∞. The lines L_1, L_2 are bitangent to C_5 and M_1 passes through Q and one of the bitangencies of L_1.

Third triple $Z_{3,\mathbb{Z}_3}$: Add a line passing through Q and a bitangency of L_2.



Results

Theorem. *Any curve described in the examples is equivalent to:*

$$Z_{1,\pm} \quad x(x-1)(y^5 - 5xy^3 + 5x^2y - 2x^3)(y^2 - \tfrac{3 \pm \sqrt{5}}{2} x) = 0$$
$$Z_{2,\pm} \quad (y^5 - 5y^3 + 5y - 2x)(x^2 - 1)(y - \tfrac{1 \pm \sqrt{5}}{2}) = 0$$

Each pair of curves forms a ***Zariski pair****. Analogously for $Z_{3,\mathbb{Z}_3}$.*

1. **Braid monodromies** can be computed from their real pictures:
   - for $Z_{1,\pm}$: $(\Delta_7, \sigma_2\sigma_3\sigma_2\sigma_6)$, $(\Delta_7, \sigma_5\sigma_6\sigma_5\sigma_3) \in \mathbb{B}_7^2$
   - for $Z_{2,\pm}$: $(\sigma_1\sigma_2\sigma_1\sigma_4, \sigma_3\sigma_5)$, $(\sigma_3\sigma_4\sigma_3\sigma_1, \sigma_2\sigma_5) \in \mathbb{B}_7^2$
2. Use Main Tool Theorem (for the Burau representation on $\mathbb{F}_3$) to check that $Z_{1,\pm}$ and $Z_{2,\pm}$ are two arithmetic Zariski pairs. These results also show that $Z_{3,\mathbb{Z}_3}$ is a Zariski triple. □

One can also study the fundamental groups of the complements:

Theorem. *Consider $G_i := \pi_1(\mathbb{C}^2 - Z_{3,i})$, $i = 1, 2, 3$. The infinite group G_3 is not isomorphic to both G_1 and G_2.*

1. Using the space of local systems on G_i, one finds for each i a unique $\rho_i : G_i \to \mathbb{Z}/10$ such that if $K_i := \ker \rho_i$ then $\operatorname{rank} K_i/K_i' = 10$.
2. $\Gamma_j(K_i)$ lower central series of K_i.
3. $\Gamma_2(G_i)/\Gamma_3(G_i) \cong \mathbb{Z}^6 \times \mathbb{Z}/5$ for $i = 1, 2$, $\Gamma_2(G_3)/\Gamma_3(G_3) \cong \mathbb{Z}^6$ □

Open Questions

Are the fundamental groups of the three arithmetic Zariski pairs non-isomorphic?

Thanks

¡¡¡Felicidades, Javier!!!

References

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