Weighted Projective Spaces

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Jun.10th-11th, 2011, ¡Felicidades Javier!

Abstract

Intersection theory is a powerful tool in complex algebraic (and analytic) geometry, see [4] for a wonderful exposition. The case of smooth surfaces is of particular interest since the intersection of objects is measured by integers. In this poster we sketch part of the intersection theory on surfaces with abelian quotient singularities and derive properties of weighted projective planes. We also use this theory to study weighted blow-ups in order to construct embedded **Q**-resolutions of plane curve singularities and abstract **Q**-resolutions of normal surfaces.

V-Manifolds and Quotient Singularities

A V-manifold of dimension n is a complex analytic space which admits an open covering $\{U_i\}$ such that U_i is analytically isomorphic to B_i/G_i where $B_i \subset \mathbb{C}^n$ is an open ball and G_i is a finite subgroup of $GL(n, \mathbb{C})$.

Example (Weighted Projective Spaces)

Let $\omega := (q_0, \ldots, q_n)$ be a weight vector, that is, a finite set of coprime positive integers. There is a natural action of the multiplicative group $\mathbb{C}^*$ on $\mathbb{C}^{n+1} \setminus \{0\}$ given by $(x_0, \ldots, x_n) \longmapsto (t^{q_0}x_0, \ldots, t^{q_n}x_n)$.
The set of orbits $\frac{\mathbb{C}^{n+1}\setminus\{0\}}{\mathbb{C}^*}$ under this action is denoted by $\mathbb{P}^n_\omega$.
Consider the decomposition $\mathbb{P}^n_\omega = U_0 \cup \cdots \cup U_n$, where U_i is the open set consisting of all elements $[x_0 : \ldots : x_n]_\omega$ with $x_i \neq 0$. The map $\widetilde{\psi}_0 : \mathbb{C}^n \longrightarrow U_0$, $\quad \widetilde{\psi}_0(x_1, \cdots, x_n) := [1 : x_1 : \ldots : x_n]_\omega$ defines an isomorphism ψ_0 if we replace $\mathbb{C}^n$ by $X(q_0; q_1, \ldots, q_n)$. Analogously, $X(q_i; q_0, \ldots, \widehat{q}_i, \ldots, q_n) \cong U_i$ under the obvious analytic map.

Remark: let $d, a_i \in \mathbb{Z}$, denote by $X(d; a_1, \ldots, a_n)$ the V-variety obtained as the quotient of $\mathbb{C}^n$ by the following action of μ_d:
$$\begin{array}{c} \mu_d \times \mathbb{C}^n \longrightarrow \mathbb{C}^n \\ (\xi_d, \mathbf{x}) \mapsto (\xi_d^{a_1} \cdot x_1, \ldots, \xi_d^{a_n} \cdot x_n). \end{array}$$

Weighted Blow-Ups and Embedded Q-Resolutions

Let X be a V-manifold with abelian quotient singularities. A hypersurface D on X is said to be with *$\mathbb{Q}$-normal crossings* if it is locally isomorphic to the quotient of a union of coordinate hyperplanes under a group action of type $(\mathbf{d}; A)$. That is, given $x \in X$, there is an isomorphism of germs $(X, x) \simeq (X(\mathbf{d}; A), [0])$ such that $(D, x) \subset (X, x)$ is identified under this morphism with a germ of the form $(\{[\mathbf{x}] \in X(\mathbf{d}; A) \mid x_1^{m_1} \cdot \ldots \cdot x_k^{m_k} = 0\}, [(0, \ldots, 0)])$.

An *embedded $\mathbb{Q}$-resolution* of $(H, 0) \subset (M, 0)$ is a proper analytic map $\pi : X \to (M, 0)$ such that:

1. X is a V-manifold with abelian quotient singularities.
2. π is an isomorphism over $X \setminus \pi^{-1}(\mathrm{Sing}(H))$.
3. $\pi^{-1}(H)$ is a hypersurface with $\mathbb{Q}$-normal crossings on X.

Classical blow-up of $\mathbb{C}^{n+1}$.

Using multi-index notation we consider
$$\widehat{\mathbb{C}}^{n+1} := \{(\mathbf{x}, [\mathbf{u}]) \in \mathbb{C}^{n+1} \times \mathbb{P}^n \mid \mathbf{x} \in \overline{[\mathbf{u}]}\}.$$
Then $\pi : \widehat{\mathbb{C}}^{n+1} \to \mathbb{C}^{n+1}$ is an isomorphism over $\widehat{\mathbb{C}}^{n+1} \setminus \pi^{-1}(0)$. The *exceptional divisor* $E := \pi^{-1}(0)$ is identified with $\mathbb{P}^n$. The space $\widehat{\mathbb{C}}^{n+1} = U_0 \cup \cdots \cup U_n$ can be covered with $n+1$ charts each of them isomorphic to $\mathbb{C}^{n+1}$. For instance, the following map defines an isomorphism:
$$\begin{array}{ccc} \mathbb{C}^{n+1} & \longrightarrow & U_0 = \{u_0 \neq 0\} \subset \widehat{\mathbb{C}}^{n+1}, \\ \mathbf{x} & \mapsto & ((x_0, x_0x_1, \ldots, x_0x_n), [1 : x_1 : \ldots : x_n]). \end{array}$$

Weighted $(p_0, \ldots, p_n)$-blow-up of $\mathbb{C}^{n+1}$.

Let $\omega = (p_0, \ldots, p_n)$ be a weight vector. As above, consider the space
$$\widehat{\mathbb{C}}^{n+1}(\omega) := \{(\mathbf{x}, [\mathbf{u}]_\omega) \in \mathbb{C}^{n+1} \times \mathbb{P}^n_\omega \mid \mathbf{x} \in \overline{[\mathbf{u}]_\omega}\}.$$
Then the natural projection $\pi : \widehat{\mathbb{C}}^{n+1}(\omega) \to \mathbb{C}^{n+1}$ is an isomorphism over $\widehat{\mathbb{C}}^{n+1}(\omega) \setminus \pi^{-1}(0)$ and the exceptional divisor $E := \pi^{-1}(0)$ is identified with $\mathbb{P}^n_\omega$. Again the space $\widehat{\mathbb{C}}^{n+1}(\omega) = U_0 \cup \cdots \cup U_n$ can be covered with $n+1$ charts. For instance the following map defines an isomorphism $\varphi_0 : X(p_0; -1, p_1, \ldots, p_n) \to U_0$ given by
$$\begin{array}{ccc} X(p_0; -1, p_1, \ldots, p_n) & \xrightarrow{\varphi_0} & U_0 = \{u_0 \neq 0\} \subset \widehat{\mathbb{C}}^{n+1}(\omega), \\ \mathbf{x} & \mapsto & ((x_0^{p_0}, x_0^{p_1}x_1, \ldots, x_0^{p_n}x_n), [1 : x_1 : \ldots : x_n]_\omega). \end{array}$$

$\mathbb{P}^2$

Local intersection number at a smooth point.

Let X be a smooth analytic surface. Consider D_1, D_2 two effective (Cartier or Weil) divisors on X and $P \in X$ a point. The divisor D_i is locally given by a holomorphic function f_i, $i = 1, 2$, in a neighborhood of P.
$$(D_1 \cdot D_2)_P = \dim_{\mathbb{C}} \left(\frac{\mathcal{O}_{X,P}}{\langle f_1, f_2 \rangle} \right).$$
Moreover, X being a smooth variety, $\mathcal{O}_{X,P}$ is isomorphic to $\mathbb{C}\{x, y\}$ and hence the previous dimension can be computed, for instance, by means of GrÃ¶bner bases with respect to local orderings.

Classical blow-up at a smooth point.

Let X be a smooth analytic surface. Let $\pi : \widehat{X} \to X$ be the classical blow-up at a (smooth) point P. Consider C and D two (Cartier. or Weil) divisors on X with multiplicities m_C and m_D at P. Denote by E the exceptional divisor of π, and by $\widehat{C}$ (resp. $\widehat{D}$) the strict transform of C (resp. D). Then,

1. $E \cdot \pi^*(C) = 0$.
2. $\pi^*(C) = \widehat{C} + m_C E$.
3. $E \cdot \widehat{C} = m_C$.
4. $E^2 = -1$.
5. $\widehat{C} \cdot \widehat{D} = C \cdot D - m_C m_D$.
6. $\widehat{D}^2 = D^2 - m_D^2$ (D compact).

Bézout's Theorem on $\mathbb{P}^2$.

The *degree of an effective divisor on* $\mathbb{P}^2$ is the degree $\deg(F)$ of the corresponding homogeneous polynomial.
Let D_1, D_2 be two divisors on $\mathbb{P}^2$,
$$\deg(D_1)\deg(D_2) = D_1 \cdot D_2 = \sum_{P \in |D_1| \cap |D_2|} (D_1 \cdot D_2)_P.$$

$\mathbb{P}^2_\omega$

Local intersection number on $X(d; a, b)$.

$$(D_1 \cdot D_2)_{[P]} = \begin{cases} \dfrac{1}{d} \dim_{\mathbb{C}} \left(\dfrac{\mathbb{C}\{x, y\}^{\mu_d}}{\langle f_1, f_2^d \rangle} \right), & P = (0,0); \\ \dim_{\mathbb{C}} \left(\dfrac{\mathbb{C}\{x - \alpha, y - \beta\}}{\langle f_1, f_2 \rangle} \right), & P = (\alpha, \beta) \neq (0,0). \end{cases}$$

Weighted blow-up

Let X be an analytic surface with abelian quotient singularities and let $\pi : \widehat{X} \to X$ be the (p, q)-weighted blow-up at a point $P \in X$ of type $(d; a, b)$. Assume $\gcd(p, q) = 1$ and $(d; a, b)$ is a normalized type, i.e. $\gcd(d, a) = \gcd(d, b) = 1$. Also write $e = \gcd(d, pb - qa)$.
Consider two $\mathbb{Q}$-divisors C and D on X. As usual, denote by E the exceptional divisor of π, and by $\widehat{C}$ (resp. $\widehat{D}$) the strict transform of C (resp. D). Let ν and μ be the (p, q)-multiplicities of C and D at P, i.e. x (resp. y) has (p, q)-multiplicity p (resp. q). Then there are the following equalities:

(1) $\pi^*(C) = \widehat{C} + \frac{\nu}{e} E$.

(2) $E \cdot \widehat{C} = \frac{e\nu}{dpq}$.

(3) $E^2 = -\frac{e^2}{dpq}$.

(4) $\widehat{C} \cdot \widehat{D} = C \cdot D - \frac{\nu\mu}{dpq}$.

In addition, if D has compact support then $\widehat{D}^2 = D^2 - \frac{\mu^2}{dpq}$.

Bézout's Theorem on $\mathbb{P}^2_\omega$

$$D_1 \cdot D_2 = \frac{1}{pqr} \deg_\omega(D_1) \deg_\omega(D_2) = \sum_{P \in |D_1| \cap |D_2|} (D_1 \cdot D_2)_P.$$

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