The dodecahedron: from intersections of quadrics to Borromean rings

Enrique ARTAL BARTOLO[*], **Santiago LÓPEZ de MEDRANO**[†] **and María Teresa LOZANO**[‡]

IUMA, Universidad de Zaragoza, 50009 Zaragoza, Spain, artal@unizar.es

Instituto de Matemáticas, Universidad Nacional Autónoma de México, 04510 México, D.F., México, santiago@matem.unam.mx

IUMA, Universidad de Zaragoza, 50009 Zaragoza, Spain, tlozano@unizar.es

Dedicado a nuestro amigo José María Montesinos Amilibia con gratitud y afecto.

ABSTRACT

The goal of this paper is to study from different points of view a manifold Z_D which is associated to a regular dodecahedron. It is a real moment-angle manifold constructed from the dodecahedron; we provide algebraic equations of this manifold using only polynomials of degree 2. This manifold has a hyperbolic manifold structure tesselated by right-angled hyperbolic dodecahedra. We study its relationship with others hyperbolic orbifolds related to the dodecahedron, and we obtain, among others results, the orbifold covering from Z_D to the hyperbolic orbifold structure on the 3-sphere with right-angled singularities at the Borromean rings.

2010 Mathematics Subject Classification: 57M10, 57M50
Key words: Orbifold, dodecahedron, orbifold covering, quadrics.

[*]Author partially supported by project MTM2013-45710-C2-1-P and DGA Grupo Consolidado E15.

[†]Author partially supported by project IN111415-PAPIIT and CEI Campus Iberus

[‡]Author partially supported by project MTM2013-45710-C2-1-P and DGA Grupo Consolidado E15.

Introduction

In the first term of the academic year 2014/15, the second named author of this paper gave a course on *Topology of intersection of quadrics* in the `Seminario de Geometrıa y Topología` organized by the IUMA at the Universidad de Zaragoza.

Among the many interesting and fruitful discussions around this course, one was related with the topological and geometric structure of the low-dimensional manifolds (or pseudo-manifolds) appearing in this context. The surfaces that appear as such intersections are well-known and classified, see Example 1.1. So, our interest turned out immediately to the 3-dimensional case.

Intersection of quadrics appear also as real moment-angle spaces associated to polytopes. For the 2-dimensional case the n-polygon is studied, while for the 3-dimensional case the starting object are polyhedra. These polyhedra have to be simple to yield smooth manifolds but the singular case is also interesting. If we restrict our attention to regular polyhedra, only the tetrahedron, the cube and the dodecahedron yield manifolds. The two first cases are easily studied, see Example 1.2, and our attention was directed to the dodecahedron D.

The manifold Z_D associated to a regular dodecahedron can be described in several ways. It is defined as an intersection of 9 diagonal quadrics in $\mathbb{R}^{12}$, see (1.3); the quotient of $Z_{\mathbf{D}}$ by the reflections along all the coordinate hyperplanes is isomorphic to $Z_D \cap \mathbb{R}^{12}_+$ which can be linearized to obtain the dodecahedron D. From this dodecahedron D, we can recover the manifold by reflecting it along all the coordinate hyperplanes. In this way the dodecahedron acquires an orbifold structure $\mathbf{D}$ with mirror faces, which is actually hyperbolic with dihedral angles equal to $\frac{\pi}{2}$ and we will denote the hyperbolic manifold as $Z_{\mathbf{D}}$. The manifold $Z_{\mathbf{D}}$ is tesselated by 2^{12} dodecahedra. This tesselation has nice symmetry properties (it is super-regular).

There is another important hyperbolic orbifold related to the dodecahedron, the universal orbifold $B_{4,4,4}$ whose underlying topological space is the 3-sphere and the $\frac{\pi}{2}$-singularities are located at the Borromean rings and whose fundamental group U has a universal property [HLMW87]. We relate $Z_{\mathbf{D}}$ and $B_{4,4,4}$ through orbifold coverings, see (3.9). As a consequence, $\pi_1(Z_{\mathbf{D}})$ is an index-2^{12} torsion-free subgroup of the universal group U.

The paper is organized as follows. In §1 we introduce the relationship between polyhedra and intersection of quadrics and as a consequence we give algebraic equations for the manifold $Z_{\mathbf{D}}$, where all of them are of degree 2. In §2 we analyze the hyperbolic structure of the orbifold $\mathbf{D}$ and the manifold $Z_{\mathbf{D}}$. This study allows to prove that $Z_{\mathbf{D}}$ is orientable, admits a super-regular tesselation and covers the well-known Löbel manifold (which itself covers $\mathbf{D}$). Finally, in §3 we relate $Z_{\mathbf{D}}$ to the universal orbifold $B_{4,4,4}$, and we relate the universal abelian covers of the above orbifolds.

We will use the following notations:

- For $n \in \mathbb{N}$, C_n is the cyclic group of order n.

- For $n \in \mathbb{N}$, Σ_n is the symmetric group of n elements.
- For $n \in \mathbb{N}$, $[n]$ will denote the set $\{1, 2, \dots, n\}$
- $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$.
- The golden ratio $\frac{1+\sqrt{5}}{2}$ will be denoted by ϕ.

1. Intersections of quadrics and polytopes

Intersection of quadrics are related to polytopes. Let A be a $k \times n$ matrix of rank k with entries in $\mathbb{R}$ and let $A_i \in \mathbb{R}^k$ be its columns. We denote by $V = V(A)$ the intersection of the quadrics in $\mathbb{R}^n$ given by the equations

$$\sum_{i=1}^{n} A_i X_i^2 = 0 \tag{1.1}$$

and by $Z = Z(A)$ the intersection of V with the unit sphere $\sum_{i=1}^{n} X_i^2 = 1$. Let also $\Pi = \Pi(A)$ be the affine subspace of $\mathbb{R}^n$ given by

$$\sum_{i=1}^{n} A_i X_i = 0, \quad \sum_{i=1}^{n} X_i = 1,$$

and $P = P(A)$ be the convex polytope $\Pi(A) \cap (\mathbb{R}_+)^n$. Both Z and P have dimension $d = n - k - 1$. The polytope P is homeomorphic to $Z \cap (\mathbb{R}_+)^n$ and, topologically, Z can be recovered from P by reflecting it in all coordinate hyperplanes.

Generically, the polytope P is transversal to all faces of $\mathbb{R}_+^n$. In that case, it is easy to see that Z is a smooth variety and P is a simple polytope[1]. In the transverse case, Z is completely determined by the combinatorics of P as a quotient of $P \times C_2^n$, while in general this may depend on the way P is embedded in $\mathbb{R}_+^n$.

It is well-known that any convex polytope of dimension d with m facets can be realized as $P(A) \subset R^n$ for any $n \geq m$ and some $k \times n$ matrix A of rank $k = n - d - 1$. Further, if the polytope is simple it can be realized transversely.

Example 1.1 Let us consider an n-polygon P_n in $\mathbb{R}_+^n$ such that each edge is the intersection with one hyperplane coordinate. The manifold Z associated to P_n is an orientable Riemann surface of genus $g_n = 2^{n-3}(n-4) + 1$, see [LdM14].

Example 1.2 There are three simple regular polytopes in dimension 3. If P is a tetrahedron it is easy to prove that the corresponding manifold Z is actually $\mathbb{S}^3 \subset \mathbb{R}^4$. Let us embed the cube Q in $\mathbb{R}^6$. We start with the symmetric cube in $\mathbb{R}^3$, i.e. the convex hull of its vertices $(\pm 1, \pm 1, \pm 1)$. The normal vectors N_i of its faces are

[1] A convex polytope of dimension d is *simple* if every one of its vertices lies in exactly d facets.

$\pm\mathbf{e}_i$ ($\mathbf{e}_i$ is the i^{th} vector of the canonical basis. Each face is in the affine plane $\{\mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} \cdot N_i = 1\}$. The affine map $\mathbb{R}^3 \to \mathbb{R}^6$ given by

$$\mathbf{x} \mapsto (X_1, \dots, X_6) = (1 - \mathbf{x} \cdot \mathbf{e}_1, 1 - \mathbf{x} \cdot \mathbf{e}_2, 1 - \mathbf{x} \cdot \mathbf{e}_3, 1 + \mathbf{x} \cdot \mathbf{e}_1, 1 + \mathbf{x} \cdot \mathbf{e}_2, 1 + \mathbf{x} \cdot \mathbf{e}_3)$$

embeds the cube Q as expected. We can check easily that Q is defined by:

$$X_i \geq 0, \quad X_1 + X_4 = 2, \quad X_2 + X_5 = 2, \quad X_3 + X_6 = 2.$$

The equations of the quadrics whose intersection is Z are:

$$X_1^2 + X_4^2 = 2, \quad X_2^2 + X_5^2 = 2, \quad X_3^2 + X_6^2 = 2.$$

From this equations is clear that Z is homeomorphic to $(\mathbb{S}^1)^3$; in fact we obtain the Clifford torus, where the induced metric structure is euclidean. We can rewrite the equations as in (1.1):

$$\sum_{i=1}^{6} X_i^2 = 6, \quad X_1^2 - X_2^2 + X_4^2 - X_5^2 = 0, \quad X_2^2 - X_3^2 + X_5^2 - X_6^2 = 0.$$

We use the sphere of radius $\sqrt{6}$ to obtain the simplest coefficients.

In the smooth case the topology of Z (and other related spaces) has been studied for sometime now ([Wal80, LdM89, LdM14, GLdM13]). Independently, in [DJ91, section 4.1], the construction of Z is given abstractly and identified as the *universal abelian cover* of P viewed as an orbifold. It is sometimes called a *real moment-angle manifold* (see footnote 2 below) and is part of a very general and abstract construction called the polyhedral product functor, see [BBCG10].

The main interest in [DJ91] is the study of other smooth covers of P of order 2^d called *small covers* which are the real topological analogs of the complex projective toric varieties[2]. It is shown there that they do not exist for all polytopes, but that they do exist for 3-dimensional simple ones[3].

In Example 1.2 we skipped the case of the dodecahedron D and its corresponding real moment-angle manifold. This manifold will be the core of this paper. The goal of the rest of this section is to give explicit and symmetric equations of this manifold as intersection of quadrics (1.1); together with the equation of a sphere centered at the origin. As we did in Example 1.2, we will take the sphere of radius $\sqrt{12}$ to obtain the simplest coefficients.

The first step is to embed the regular dodecahedron in $\mathbb{R}_+^{12}$ such that each face F_i is the intersection with the coordinate hyperplane $\{X_i = 0\}$. We number the faces F_i

[2]There are other related manifolds with torus actions called *moment-angle* manifolds and *(quasi)-toric* manifolds

[3]The proof of this fact uses the Four Color Theorem!

of the dodecahedron as in Figure 1. We start with the regular dodecahedron in $\mathbb{R}^3$, whose vertices v_k are

$$\{(\pm 1, \pm 1, \pm 1)\} \cup \bigcup_{j=0}^{2} \{(0, \pm\phi, \pm\phi^{-1})\}^{\rho^j}$$

where ρ stands for the right action $(x, y, z)^\rho := (z, x, y)$. This is the dodecahedron inscribed in the sphere of radius $\sqrt{3}$. The normal vectors N_j of the faces are all the cyclic permutations of $(\pm(2-\phi), \pm(\phi-1), 0)$ ordered as in Figure 1 (we leave to the reader to give an explicit ordering). If a vertex is in a face, it satisfies $N_j \cdot v_k = 1$. The embedding of D in $\mathbb{R}^{12}$ is given by the affine map $\mathbb{R}^3 \to \mathbb{R}^{12}$:

$$(x, y, z) \mapsto (1 - (x, y, z) \cdot N_1, \ldots, 1 - (x, y, z) \cdot N_{12}).$$

The group of isometries of the dodecahedron (as permutation of the faces) is generated by the following permutations of the coordinates in $\mathbb{R}^{12}$:

1. Central symmetry: $(1\,12)(2\,11)(3\,10)(4\,9)(5\,8)(6\,7)$.

2. Rotation of angle $\frac{2\pi}{5}$ around the axis joining the centers of faces 1 and 12: $(2\,3\,4\,5\,6)(7\,11\,10\,9\,8)$

3. Rotation of angle $\frac{2\pi}{3}$ around the axis joining the vertices $(1, 2, 6)$ and $(7, 11, 12)$: $(1\,2\,6)(3\,9\,5)(4\,8\,10)(7\,12\,11)$.

4. Rotation of angle π around the axis joining the centers of the edges $(1, 2)$ and $(11, 12)$: $(1\,2)(3\,6)(4\,9)(5\,8)(7\,10)(11\,12)$.

One of the vertices is:

$$\frac{1}{6}\left(0,\, 0,\, 0,\, 2-\phi,\, \phi-1,\, 2-\phi,\, \phi-1,\, 2-\phi,\, \phi-1,\, 1,\, 1,\, 1\right), \tag{1.2}$$

and the other ones are obtained using the symmetry group.

Equalities involving normal vectors N_i imply equations satisfied by the points in the dodecahedron in $\mathbb{R}^{12}$. They are:

1. $N_i + N_{13-i} = 0$, i.e. $X_i + X_{13-i} = 2$. Adding up, we obtain $\sum_{i=1}^{12} X_i = 12$ and the following system of 15 homogeneous $\{X_i + X_{13-i} - X_j - X_{13-j} = 0 \mid 1 \leq i < j \leq 6\}$ which are invariant by the above group of permutations (eventually a permutation can change the sign of the equation).

   Note that only 5 of these equations are linearly independent (say $(i, j) = (1, 2), (2, 3), (3, 4), (4, 5), (5, 6)$).

2. Consider a face, say the first one, and the five neighboring faces. We obtaing the equality $N_2 + \cdots + N_6 - \sqrt{5}N_1 = 0$. Hence, we have:

$$X_2 + \cdots + X_6 - \sqrt{5}X_1 = 5 - \sqrt{5}.$$

As before, we can consider the homogeneous equations associated to a pair of faces. In fact, we can restrict our attention to the equations (up to sign) associated to a pair of adjacent faces, i.e., to edges, denoted as $i \asymp j$. If we denote an edge by its neighboring faces $\{i, j\}$, we have 30 equations:

$$\left\{ \sum_{k \asymp i} X_k - \sqrt{5}X_i = \sum_{\ell \asymp j} X_\ell - \sqrt{5}X_j \mid \{i, j\} \text{ edge} \right\}.$$

These 30 equations contain 8 independent equations, e.g., consider the equations associated to a set E formed by four edges of F_1 and four edges of F_2 such that the forgotten edges are not opposite. Finally, we can give a family of 9 equations of the real moment-angle manifold associated to the dodecahedron:

$$\sum_{i=1}^{12} X_i^2 = 12, \quad \left\{ \sum_{k \asymp i} X_k^2 - \sqrt{5}X_i^2 = \sum_{\ell \asymp j} X_\ell^2 - \sqrt{5}X_j^2 \mid \{i, j\} \in E \right\}. \tag{1.3}$$

2. Orbifold covering spaces of the right angle dodecahedron

In the previous section we studied some properties of the regular dodecahedron $D \subset \mathbb{R}^3$ and of its embedding in $\mathbb{R}^{12}$. In this section we study the hyperbolic regular dodecahedron $\mathbf{D}$ such that every dihedral angle defined by two adjacent faces is $\frac{\pi}{2}$. The hyperbolic space H^3 has a super regular tessellation T by copies of $\mathbf{D}$ [Thu80]. The group $G_{\mathbf{D}}$ generated by reflection on the twelve planes containing the faces of any dodecahedron acts on H^3 preserving the tessellation and has the dodecahedron as fundamental domain. The quotient $H^3/G_{\mathbf{D}}$ defines a hyperbolic orbifold structure in $\mathbf{D}$ with mirror singularity on every face. Then the reflection group $G_{\mathbf{D}}$ is the orbifold fundamental group of $\mathbf{D}$. In order to understand the group $G_{\mathbf{D}}$ we number the faces of $\mathbf{D}$ as in Figure 1. Let x_i be the reflection on the face $\mathbf{i}$. If $\mathbf{i}$ and $\mathbf{j}$ are two adjacent faces, denote by l_{ij} the common edge. The vertex v_{ijk} is the common vertex of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ when they intersect. The group $G_{\mathbf{D}}$ is generated by $x_1, ..., x_{12}$. The isotropy subgroups $H(\bullet)$ of the singular elements are given by the following generators and relations:

$$\begin{array}{lcl} H(\mathbf{i}) & = & |x_i;\, x_i^2| = C_2 \\ H(l_{ij}) & = & |x_i, x_j;\, x_i^2, x_j^2, (x_i x_j)^2| = C_2 \times C_2 \\ H(v_{ijk}) & = & |x_i, x_j, x_k;\, x_i^2, x_j^2, x_k^2, (x_i x_j)^2, (x_j x_k)^2, (x_k x_i)^2| = C_2 \times C_2 \times C_2 \end{array}$$

Hence,

$$G_{\mathbf{D}} = \left| x_1, ..., x_{12} : \underset{1 \le i \le 12}{x_i^2}, \quad \underset{l_{ij} \text{ edge}}{(x_i x_j)^2} \right|.$$

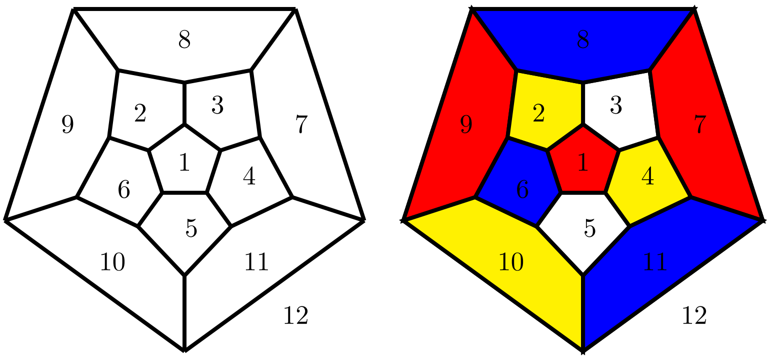

Figure 1: Dodecahedron **D** and one coloring on **D**

The orbifold coverings over **D** are classified by the conjugation classes of subgroups of $G_{\mathbf{D}}$. To construct a n-fold covering, we should define a monodromy map on the permutation group of n elements:

$$\omega_n : G_{\mathbf{D}} \longrightarrow \Sigma_n$$

If each permutation $\omega_n(x_i)$, $i = 1, ..., 12$, is the product of $\frac{n}{2}$ different transpositions, then the cover has no mirror singular set. Observe that in this case the number of sheets should be even. The minimal number of sheets to obtain a manifold is at least 8 because the isotropy subgroup of each vertex of **D** for the action of $G_{\mathbf{D}}$, is isomorphic to the 8-element abelian group C_2^3. Therefore one needs 8 copies of the fundamental domain around the preimage of each vertex to kill the singularity. For $n = 2, 4, 6$ the cover has singularities. For instance, the case $n = 2$ produces the double of **D**, that is, an orientable hyperbolic orbifold structure in S^3 with a singular trivalent graph, the 1-skeleton of the dodecahedron, where the angle around the edges is π.

2.1. The Löbell manifold

The Löbell manifold $L(5)$ constructed in [Ves98], see also [Löb31], is an $8:1$ cover of **D** defined by the coloring depicted in Figure 1. The procedure to construct the monodromy of the covering is the following: Color the faces with 4 different colors (R, Y, B, W) such that the three faces meting in a vertex have different color. Then define a map sending the generator x_i to an element of C_2^3 according to the assigned color as follows. Let

$$C_2 \times C_2 \times C_2 \equiv (C_2\langle R\rangle \oplus C_2\langle Y\rangle \oplus C_2\langle B\rangle \oplus C_2\langle W\rangle) \,/\, (C_2\langle R+Y+B+W\rangle)\,.$$

The map ρ_L

$$\begin{aligned}
\rho_L : G_{\mathbf{D}} &\longrightarrow C_2 \times C_2 \times C_2\\
x_1, x_7, x_9 &\longmapsto R \equiv (1,0,0)\\
x_2, x_4, x_{10} &\longmapsto Y \equiv (0,1,0)\\
x_6, x_8, x_{11} &\longmapsto B \equiv (0,0,1)\\
x_3, x_5, x_{12} &\longmapsto B \equiv (1,1,1)
\end{aligned}$$

is a homomorphism because each generator goes to an order two element and the group C_2^3 is abelian. On the other hand, the three different colors at each vertex is a necessary and sufficient condition in order to have a manifold.

Now number the eight elements of $C_2 \times C_2 \times C_2$, for instance:

$$\begin{array}{llll}
(1,0,0)=1 & (0,0,0)=2 & (0,1,0)=3 & (1,1,0)=4\\
(0,0,1)=5 & (1,0,1)=6 & (0,1,1)=7 & (1,1,1)=8
\end{array}$$

and define the monodromy as the permutation associated to the left action of $\rho(x_i)$ onto C_2^3:

$$\begin{aligned}
\omega_L : G_{\mathbf{D}} &\longrightarrow \Sigma_8\\
x_1, x_7, x_9 &\longmapsto (12)(34)(56)(78)\\
x_2, x_4, x_{10} &\longmapsto (14)(23)(57)(68)\\
x_6, x_8, x_{11} &\longmapsto (16)(25)(37)(48)\\
x_3, x_5, x_{12} &\longmapsto (17)(28)(36)(45)
\end{aligned}$$

This monodromy ω_L define the 8-fold orbifold covering

$$p_{L(5)} : L(5) \xrightarrow{8:1} \mathbf{D}$$

The fundamental group of the manidold $L(5)$ is the preimage by ω_L of the stabilizer in Σ_8 of one element, for instance 1. By the above construction of ω_L it is easy to see that this subgroup is the kernel of the homomorphism ρ_L. The kernel is a normal subgroup, then the orbidold covering is regular and the group of deck transformations is C_2^3.

This procedure can be generalized to construct the monodromy of some 2^s-fold coverings ($4 \le s \le 12$) as follows: The number n of different colors on the faces should be 4, 6 or 12 colors such that the three faces meting in a vertex have different color and the number of faces of the same color coincides (3, 2 or 1). The map $\rho : G_{\mathbf{D}} \longrightarrow C_2^s$ should be onto, therefore $n \ge s$.

2.2. The abelian universal covering

We are interested in one of these coverings, the one with more sheets, the abelian universal covering. Here $s = n = 12$. Observe that the homomorphism ρ_u is the abelianization of the group $G_{\mathbf{D}}$.

$$\begin{aligned} \rho_u : G_{\mathbf{D}} &\longrightarrow (C_2)^{12} \\ x_i &\longmapsto (0, \ldots, \underset{(i)}{1}, \ldots, 0) \end{aligned}$$

Then the kernel of ρ_u is the commutator subgroup or derived subgroup $G'_{\mathbf{D}}$, which is the smallest normal subgroup such that the quotient group of the original group $G_{\mathbf{D}}$ by this subgroup is abelian.

As before, the action of $\rho_u(x_i)$ on the 2^{12} elements of $(C_2)^{12}$ defines the monodromy

$$\omega_u : G_{\mathbf{D}} \longrightarrow \Sigma_{2^{12}}$$

of the regular orbifold covering

$$p_u : Z_{\mathbf{D}} \xrightarrow{2^{12}:1} \mathbf{D}$$

The fundamental group of the hyperbolic manifold $Z_{\mathbf{D}}$ is the derived subgroup $G'_{\mathbf{D}}$ and the group of deck transformations is $(C_2)^{12}$.

Theorem 2.1 *The hyperbolic manifold $Z_{\mathbf{D}}$ is a 2^9-fold covering of $L(5)$.*

Proof. The homomorphism ρ_L factors though the homomorphism ρ_u as follows:

$$\begin{tikzcd} G_{\mathbf{D}} \arrow[rr, "\rho_L"] \arrow[dr, "\rho_u"'] & & (C_2)^3 \\ & (C_2)^{12} \arrow[ur, "h"'] & \end{tikzcd}$$

□

2.3. The super regular tessellation of $Z_{\mathbf{D}}$

The hyperbolic manifold $Z_{\mathbf{D}}$ has a super regular tessellation by dodecahedra, because the 2-skeleton and the 1-skeleton also have a regular tessellation by hyperbolic pentagons and geodesics respectively. Next we analyze these submanifolds.

Let l be an edge of the dodecahedron. Then $p_u^{-1}(l)$ is the disjoint union of 2^8 closed geodesics composed by 4 segments of the same length. Therefore the 1-skeleton of $Z_{\mathbf{D}}$ is a 6-valent graph composed by the union of $30 \times 2^8 = 15 \times 2^9$ isometric closed hyperbolic geodesics arranged in 30 families, such that two geodesics in the same family do not intersect, there are exactly 5×2^{11} triple intersection points, and every closed geodesic has 4 of them.

Let P be a face of the dodecahedron. Then $p_u^{-1}(P)$ is the disjoint union of $2^6 = 64$ hyperbolic surfaces of genus 5, F_5. The 2-skeleton of $Z_{\mathbf{D}}$ is the union of $12 \times 2^6 = 3 \times 2^8$ hyperbolic surfaces F_5 arranged in 12 families, such that two surfaces in the same family do not intersect. If two surfaces intersect they do it along a closed geodesic and the surfaces correspond to two adjacent pentagons in the fundamental dodecahedron. Then every surface intersects with surfaces in five other families.

Theorem 2.2 *Every F_5 surface in the preimage by p_u of a face of the dodecahedron is a embedded incompressible surface in $Z_{\mathbf{D}}$.*

Proof. A orientable surface embedded in a manifold is incompressible if the fundamental group of the surface injects in the fundamental group of the manifold. Suppose that F_5 is the surface in the preimage of the face **1** generated by the reflection on the five adjacent pentagons **2**, **3**, **4**, **5** and **6**, see Figure 1. The group $G_{\mathbf{1}}$ of the orbifold structure in **1** is the subgroup of $G_{\mathbf{D}}$ generated by the reflection $(x_2, x_3, x_4, x_5, x_6)$ and F_5 is the abelian universal covering of the orbifold **1**. Then the following diagram is commutative.

$$\begin{array}{ccc} G_{\mathbf{1}} & \xrightarrow{\rho_{u1}} & (C_2)^5 \\ \downarrow & & \downarrow \\ G_{\mathbf{D}} & \xrightarrow[\rho_u]{} & (C_2)^{12} \end{array}$$

where the vertical maps are inclusions. Therefore the fundamental group of F_5, being the kernel of ρ_{u1}, injects in the kernel of ρ_u, which is the fundamental group of $Z_{\mathbf{D}}$. □

The volume of **D** has been computed in [Ves98] (see also [Ves10, Theorem 3.2]).

Corollary 2.3 *The volume of $Z_{\mathbf{D}}$ is $2^{12}\,\mathrm{Vol}(\mathbf{D}) = 2^9\,\mathrm{Vol}(L(5))$.*

3. The orbifolds D and $B_{4,4,4}$

The hyperbolic orbifold structure $B_{4,4,4}$ in S^3 with singular set the Borromean rings with cyclic isotropy group of order 4 is the quotient of the hyperbolic space H^3 by the universal group U [HLMW87] and has also a regular right angle hyperbolic dodecahedron as fundamental domain ([Thu80]), see Figure 2.

Here is a presentation of U, the fundamental group of the orbifold $B_{4,4,4}$:

$$\begin{aligned} U = \langle a, b, c \mid a\, b\bar{c}\bar{b}c &= b\bar{c}\bar{b}c\, a,\ a^4, \\ b\, c\bar{a}\bar{c}a &= c\bar{a}\bar{c}a\, b,\ b^4, \\ c\, a\bar{b}\bar{a}b &= a\bar{b}\bar{a}b\, c,\ c^4 \rangle \end{aligned} \tag{2.1}$$

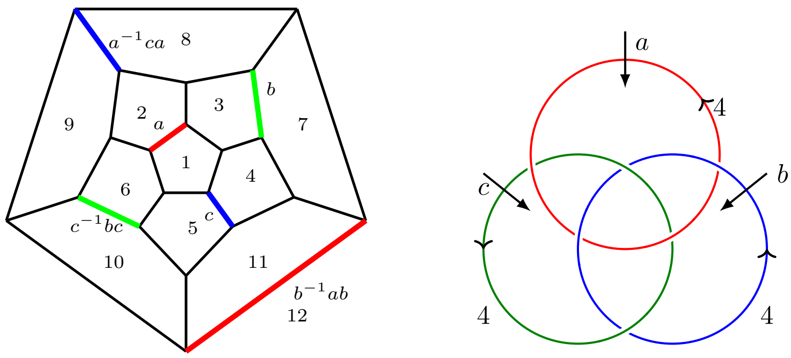

Figure 2: The orbifold $B_{4,4,4}$

The generators a, b, and c arise from the three meridian generators for the three components of the Borromean rings and they are $\frac{\pi}{2}$-rotation around the corresponding edge of the dodecahedron.

The fundamental orbifold group of $B_{4,4,4}$ is U. Therefore both groups, $G_{\mathbf{D}}$ and U act on the hyperbolic space H^3 by isometries fixing the same tessellation T by regular right angle hyperbolic dodecahedra and having the same fundamental domain. Then the two hyperbolic orbifolds $\mathbf{D}$ and $B_{4,4,4}$ have the same volume. They are related as follows.

Depicted in Figure 3 is the intersection of the right angle hyperbolic dodecahedron centered in the origin of coordinates with the positive octant. Let us call $\mathbf{Q}$ the orbifold structure on this hyperbolic polyhedron with mirror faces. The dihedral angle are all right angles, excepted the colored a, b end c which are $\frac{\pi}{4}$. The orbifold fundamental group is generated by the reflection on its six numerated faces. Let us call y_i the reflection on the i face.

Lemma 3.1 *The orbifold* $\mathbf{D}$ *and the orbifold* $B_{4,4,4}$ *are eight-fold regular orbifold covers of the orbifold* $\mathbf{Q}$.

Proof. The monodromy for the covering $q_{\mathbf{D}} : \mathbf{D} \longrightarrow \mathbf{Q}$ is obtained using the homomorphism

$$\begin{aligned} \rho_{DQ} : G_{\mathbf{Q}} &\longrightarrow C_2 \times C_2 \times C_2 \\ y_1 &\longmapsto (1,0,0) \\ y_2 &\longmapsto (0,1,0) \\ y_3 &\longmapsto (0,0,1) \\ y_4, y_5, y_6 &\longmapsto (0,0,0) \end{aligned} \tag{3.1}$$

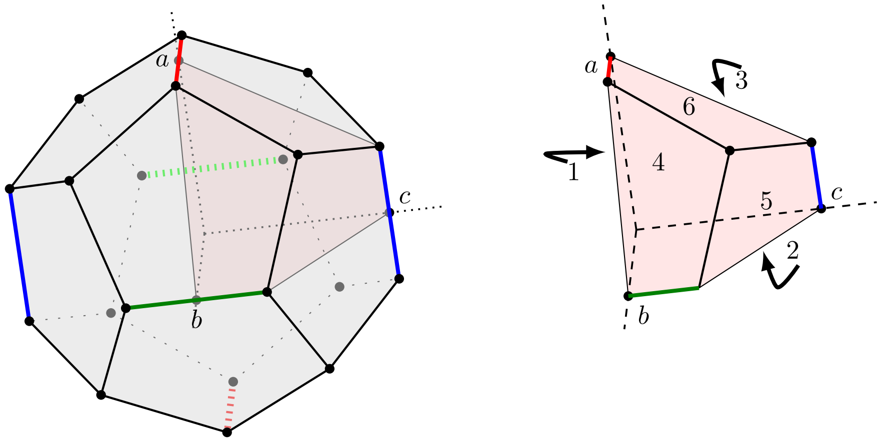

Figure 3: The orbifoldS $\mathbb{D}$ and $\mathbf{Q}$.

The monodromy for the covering $q_B : B_{4,4,4} \longrightarrow \mathbf{Q}$ is obtained using the homomorphism

$$\begin{aligned} \rho_{BQ} : G_{\mathbf{Q}} &\longrightarrow C_2 \times C_2 \times C_2 \\ y_1, y_6 &\longmapsto (1,0,0) \\ y_2, y_4 &\longmapsto (0,1,0) \\ y_3, y_5 &\longmapsto (0,0,1) \end{aligned} \tag{3.2}$$

□

Therefore the fundamental groups $G_{\mathbf{D}}$ and U of the orbifolds $\mathbf{D}$ and $B_{4,4,4}$ are index eight subgroups of $G_{\mathbf{Q}}$. Let us recall the concept of commensurable subgroups. Two subgroups A and B of a group are *commensurable* when their intersection has finite index in each of them. This property is an equivalence relation. Therefore, $G_{\mathbf{Q}}$, $G_{\mathbf{D}}$ and U are commensurable subgroups. But we can analyze more commensurable subgroups of $G_{\mathbf{Q}}$.

The universal abelian cover $p_{uQ} : Z_{\mathbf{Q}} \xrightarrow{2^6} \mathbf{Q}$ is associated to the abelianization homomorphism

$$\begin{aligned} \rho_{uQ} : G_{\mathbf{Q}} &\longrightarrow C_2^6 \\ y_i &\longmapsto (0,\ldots,0,\underset{(i)}{1},0,\ldots,0) \end{aligned} \tag{3.3}$$

Observe that $Z_{\mathbf{Q}}$ is a compact hyperbolic orbifold with singular set composed by closed geodesics of order 2, corresponding to the edges of $\mathbf{Q}$ with angle $\frac{\pi}{4}$.

The universal abelian cover $p_{uB} : Z_B \xrightarrow{2^6} B_{4.4,4}$ is associated to the abelianization homomorphism

$$\begin{aligned} \rho_{uB} : U &\longrightarrow C_4^3 \\ a &\longmapsto (1,0,0) \\ b &\longmapsto (0,1,0) \\ c &\longmapsto (0,0,1) \end{aligned} \tag{3.4}$$

Here Z_B is a compact hyperbolic manifold, all singularities are removed.

The abelian covers are associated to the derived group of the fundamental group of the base orbifold. The commutative diagram (3.5) relates those groups and their derived subgroups. All the maps are inclusions and the label is the index of the subgroup.

$$\begin{tikzcd} G'_{\mathbf{D}} \arrow[rr, "2^{12}"] \arrow[dr, "2^9"] & & G_{\mathbf{D}} \arrow[dr, "2^3"] & \\ & G'_{\mathbf{Q}} \arrow[rr, "2^6"] & & G_{\mathbf{Q}} \\ U' \arrow[ur, "2^3"] \arrow[rr, "4^3"] & & U \arrow[ur, "2^3"] & \end{tikzcd} \tag{3.5}$$

This diagram has a counter-part for covering maps:

$$\begin{tikzcd} Z_{\mathbf{D}} \arrow[rr, "2^{12}"] \arrow[dr, "2^9"] & & \mathbf{D} \arrow[dr, "2^3"] & \\ & Z_{\mathbf{Q}} \arrow[rr, "2^6"] & & \mathbf{Q} \\ Z_B \arrow[ur, "2^3"] \arrow[rr, "4^3"] & & B_{4,4,4} \arrow[ur, "2^3"] & \end{tikzcd}$$

It is natural to look for a common finite cover of the two hyperbolic orbifolds $\mathbf{D}$ and $B_{4,4,4}$. For instance, the smaller one is the covering associated to the intersection of their fundamental groups $H = G_{\mathbf{D}} \cap U$.

Lemma 3.2 *The index of the subgroup H in $G_{\mathbf{D}}$ and U is eight.*

Proof. The subgroup H is the kernel of the surjective homomorphism

$$(\rho_{DQ}, \rho_{BQ}) : G_{\mathbf{Q}} \longrightarrow (C_2)^3 \times (C_2)^3 \tag{3.6}$$

where ρ_{DQ} and ρ_{BQ} are defined in (3.1) and (3.2). Observe that (ρ_{DQ}, ρ_{BQ}) is equal to ρ_{uQ}, see (3.3). The kernel of this homorphism is $G'_{\mathbf{Q}} = H$, has index 2^6 in $G_{\mathbf{Q}}$ and is contained in the index 2^3 subgroups $G_{\mathbf{D}}$ and U which have index 2^3. Therefore the index of H in $G_{\mathbf{D}}$ and U is $2^3 = 8$. $\square$

Theorem 3.3 *The orbifold $Z_{\mathbf{Q}}$ is the minimal common orbifold covering of the hyperbolic orbifolds $\mathbf{D}$ and $B_{4,4,4}$.*

Proof. The homomorphism ρ_{BQ} factors through $\rho_{uQ} = (\rho_{DQ}, \rho_{BQ})$ and so does ρ_{DQ}. Therefore the covering $p_{uQ} : Z_{\mathbf{Q}} \xrightarrow{2^6} \mathbf{Q}$ factors through $q_B : B_{4,4,4} \longrightarrow \mathbf{Q}$ and also through $q_{\mathbf{D}} : \mathbf{D} \longrightarrow \mathbf{Q}$.

$$\begin{tikzcd}
& Z_{\mathbf{Q}} \arrow[dl, "2^3:1"', "p_1"] \arrow[dd, "\rho_{uQ}"', "2^6:1"] \arrow[dr, "2^3:1", "p_2"'] & \\
B_{4,4,4} \arrow[dr, "q_B", "2^3:1"'] & & \mathbf{D} \arrow[dl, "q_{\mathbf{D}}"', "2^3:1"] \\
& \mathbf{Q} &
\end{tikzcd}$$

In fact, we can construct directly the coverings $p_1 : Z_{\mathbf{Q}} \longrightarrow B_{4,4,4}$ and $p_2 : Z_{\mathbf{Q}} \longrightarrow \mathbf{D}$ as follows. The subgroup H_1 generated by the following elements is a subgroup of $H = G_{\mathbf{D}} \cap U$:

$$\begin{array}{lll}
a^2 = x_1x_2, & b^2 = x_3x_9, & c^2 = x_4x_5, \\
b^{-1}a^2b = x_{10}x_{12}, & c^{-1}b^2c = x_6x_{11}, & a^{-1}c^2a = x_7x_7
\end{array}$$

The homomorphim

$$\begin{aligned}
\rho : U &\longrightarrow C_2 \times C_2 \times C_2 \\
a &\longmapsto (1,0,0) \\
b &\longmapsto (0,1,0) \\
c &\longmapsto (0,0,1)
\end{aligned} \tag{3.7}$$

defines the monodromy

$$\begin{aligned}
\omega_1 : U &\longrightarrow \Sigma_8 \\
a &\longmapsto (1\,2)(3\,4)(5\,6)(7\,8) \\
b &\longmapsto (1\,4)(2\,3)(5\,7)(6\,8) \\
c &\longmapsto (1\,6)(2\,5)(3\,6)(4\,8)
\end{aligned}$$

It defines the regular 8-fold orbifold covering of $B_{4,4,4}$

$$p_1 : Z_{\mathbf{Q}} \longrightarrow B_{4,4,4}$$

The map p_1 can be viewed in different ways:

- An 8-fold locally cyclic covering of the sphere S^3 branched over the Borromean rings with branching index 2.

- An 8-fold orbifold covering of $B_{2,2,2}$, where $B_{2,2,2}$ is the Euclidean orbifold structure in S^3 with singular set de Borromean ring with cyclic isotropy group of order 2.

This implies that $Z_{\mathbf{Q}}$ has a Euclidean manifold structure, with no singularity and also a hyperbolic orbifold structure with a singular link. Note that the monodromy in (3.2) factors through the one in (3.4).

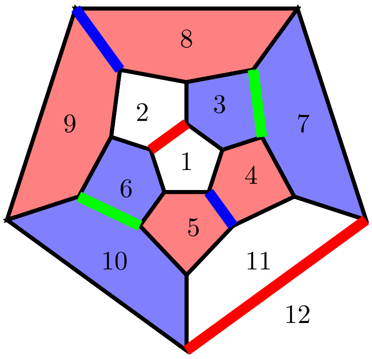

Figure 4: Color on $\mathbf{D}$ for the minimal common covering of $\mathbf{D}$ and $B_{4,4,4}$

On the other hand, we can color the dodecahedron with three colors as in Figure 4. The colors define the homomorphism

$$\begin{aligned} \rho : G_{\mathbf{D}} &\longrightarrow C_2 \times C_2 \times C_2 \\ x_1, x_2, x_{11}, x_{12} &\longmapsto (1,0,0) \\ x_4, x_5, x_8, x_9 &\longmapsto (0,1,0) \\ x_3, x_7, x_6, x_{10} &\longmapsto (0,0,1) \end{aligned} \tag{3.8}$$

and the monodromy

$$\begin{aligned} \omega_2 : G_{\mathbf{D}} &\longrightarrow \Sigma_8 \\ x_1, x_2, x_{11}, x_{12} &\longmapsto (1\,2)(3\,4)(5\,6)(7\,8) \\ x_4, x_5, x_8, x_9 &\longmapsto (1\,4)(2\,3)(5\,7)(6\,8) \\ x_3, x_7, x_6, x_{10} &\longmapsto (1\,6)(2\,5)(3\,6)(4\,8) \end{aligned}$$

defines the orbifold covering

$$p_2 : Z_{\mathbf{Q}} \longrightarrow \mathbf{D}$$

Observe that the three different colors on the three faces around a vertex implies that there are no singularities at the preimage of the vertex and the edges. The singular link corresponds to the preimage of the colored edges common to two faces with the same color. □

Remark 3.1 Let $\mathbf{Q}_2$ be the geometric orbifold structure on $\mathbf{Q}$ where all the angles are $\frac{\pi}{2}$, i.e., $\mathbf{Q}_2$ is the euclidean cube. The universal abelian cover $Z_{\mathbf{Q}_2}$ can be obtained also as a real moment-angle manifold, or an intersection of quadrics in $\mathbb{R}^6$, see Example 1.2, where we showed that it is homeomorphic to the three-torus $(\mathbb{S}^1)^3$ (see e.g. [LdM14]). Topologically $Z_{\mathbf{Q}} \equiv Z_{\mathbf{Q}_2}$ and this proves also that this topological manifold has the two following geometric structures: the euclidean one (with no singularities) and a hyperbolic one (with 12 circles with angle π).

As a consequence, the manifold $Z_{\mathbf{D}}$ is a 2^9-fold cover of the three-torus branched over a 12-component link.

Corollary 3.4 *The minimal common orbifold covering of the hyperbolic orbifolds* $\mathbf{D}$ *and* $B_{4,4,4}$ *which is a hyperbolic manifold is a two-fold covering of* $Z_{\mathbf{Q}}$*, made up with 16 dodecahedra.*

$$K \xrightarrow{2:1} Z_{\mathbf{Q}}$$

□

Corollary 3.5 *The Löbell manifold* $L(5)$ *is not an orbifold covering of the orbifold* $B_{4,4,4}$. □

Proposition 3.6 *The orbifold covering* $p_1 : Z_{\mathbf{Q}} \longrightarrow B_{4,4,4}$ *factors through a 4-fold orbifold covering* $p_3 : N \longrightarrow B_{4,4,4}$

Proof. The orbifold covering $p_3 : N \longrightarrow B_{4,4,4}$ is associated to the monodromy

$$\begin{aligned} \omega_3 : U &\longrightarrow \Sigma_4 \\ a &\longmapsto (1\,2)(3\,4) \\ b &\longmapsto (1\,3)(2\,4) \\ c &\longmapsto (1\,4)(2\,3) \end{aligned}$$

The cover N is a hyperbolic orbifold with a singular link of order 2. The fundamental group of the orbifold N is the kernel of ω_3, so it contains H_1 and has index 4 in U. The map p_3 is also a 4-fold locally cyclic covering of the sphere S^3 branched over the Borromean rings with branching index 2, or a 4-fold orbifold covering of $B_{2,2,2}$. The topological space N has both a Euclidean manifold structure (no singularity) and a hyperbolic orbifold structure with a singular link.

$$\begin{tikzcd} Z_{\mathbf{Q}} \arrow[rr, "2:1"] \arrow[dr, "2^3:1", "p_1"'] & & N \arrow[dl, "2^2:1"', "p_3"] \\ & B_{4,4,4} & \end{tikzcd}$$

□

Proposition 3.7 *The orbifold covering $p_2 : Z_{\mathbf{Q}} \longrightarrow \mathbf{D}$ factors through a 4-fold orbifold covering $p_4 : M \longrightarrow \mathbf{D}$*

Proof. The orbifold covering $p_4 : M \longrightarrow \mathbf{D}$ is associated to the monodromy

$$\begin{aligned} \omega_4 : G_{\mathbf{D}} &\longrightarrow \Sigma_4 \\ x_1, x_2, x_{11}, x_{12} &\longmapsto (1\,2)(3\,4) \\ x_4, x_5, x_8, x_9 &\longmapsto (1\,3)(2\,4) \\ x_3, x_7, x_6, x_{10} &\longmapsto (1\,4)(2\,3) \end{aligned}$$

The cover M is a hyperbolic orbifold with a singular link of order 2. The fundamental group of the orbifold M is the kernel of ω_4, so it contains H_1 and has index 4 in $G_{\mathbf{D}}$.

$$\begin{tikzcd} Z_{\mathbf{Q}} \arrow[rr, "2:1"] \arrow[dr, "2^3:1", "p_2"'] & & M \arrow[dl, "2^2:1"', "p_4"] \\ & \mathbf{D} & \end{tikzcd}$$

□

Proposition 3.8 *The 2-fold orbifold covering $p_5 : M_1 \longrightarrow M$ which is the 2-fold covering of M branched over the singular link, is the Löbell hyperbolic manifold $L(5)$.*

Proof. The hyperbolic orbifold cover is actually a hyperbolic manifold because all the branching index are 2 and the singular link in M have order 2. There are no singularities in M_1. Then $p_4 \circ p_5 : M_1 \longrightarrow \mathbf{D}$ is an 8-fold orbifold covering. It is a *small cover* of $\mathbf{D}$ ([DJ91]). It is proved in [GS03] that $\mathbf{D}$ has 25 small covers, but the only orientable one is the Löbell manifold $L(5)$. Then $M_1 = L(5)$. □

The following diagram summarizes all the relations among the studied orbifold coverings between the hyperbolic manifold $Z_{\mathbf{D}}$ and the orbifold $\mathbf{Q}$.

$$\begin{tikzcd} & & L(5) \arrow[r, "2"] & M \arrow[r, "2^2"] & \mathbf{D} \arrow[dr, "2^3"] & \\ Z_{\mathbf{D}} \arrow[r, "2^8"] & K \arrow[ur] \arrow[r, "2"] & Z_{\mathbf{Q}} \arrow[ur, "2"'] \arrow[dr, "2"] & & & \mathbf{Q} \\ & & & N \arrow[r, "2^2"'] & B_{4,4,4} \arrow[ur, "2^3"'] & \end{tikzcd} \tag{3.9}$$

This diagram has a counter-part for the fundamental orbifold groups. Here all the maps are inclusions.

$$
\begin{tikzcd}
 & & G_L \arrow[r, "2"] & \ker\omega_4 \arrow[r, "2^2"] & G_{\mathbf{D}} \arrow[dr, "2^3"] & \\
G'_{\mathbf{D}} \arrow[r, "2^8"] & G_K \arrow[r, "2"] \arrow[ur] & G'_{\mathbf{Q}} \arrow[ur, "2"'] \arrow[dr, "2"] & & & G_{\mathbf{Q}} \\
 & & & \ker\omega_3 \arrow[r, "2^2"'] & U \arrow[ur, "2^3"'] &
\end{tikzcd}
\tag{3.10}
$$

Theorem 3.9 *All the groups in (3.10) and* (3.5) *are commensurable subgroups of* $G_{\mathbf{Q}}$ *and they are arithmetic.*

Proof. The groups in (3.10) and (3.5) are part of the lattice of subgroups of the group $G_{\mathbf{Q}}$. All the inclusions in the diagram have finite index. Then all of them are commensurable subgroups. In [HLM92] the arithmeticity of the groups was studied for hyperbolic structures on the octant of the dodecahedron with different angles around the colored edges (Figure 3). One of then, $R(4,4,4)$, is the orbifold $\mathbf{Q}$. It is proved there that $\mathbf{Q} = R(4,4,4)$ is arithmetic. Therefore all the subgroups in (3.10) are arithmetic. In fact the concept of arithmetic subgroup is related with the problem of enumerating all the *forms of Clifford-Klein* or *geometric orbifolds of constant curvature, complete and with finite volume*, in actual language, as was pointed out in the historical comments contained in [Mon13]. $\square$

It is possible to compute all the groups in (3.10) but most of them have very long presentations. For example, $\pi_1(Z_{\mathbf{D}}) = G'_{\mathbf{D}}$ can be computed using `Sagemath` [S+15] and `GAP4` [GAP15]. We obtain a group whose abelianization is a free abelian group of rank 935. It is possible to find a presentation with 935 generators and 955 relations (all of them product of commutators).

References

[BBCG10] A. Bahri, M. Bendersky, F. R. Cohen, and S. Gitler, *The polyhedral product functor: a method of decomposition for moment-angle complexes, arrangements and related spaces*, Adv. Math. **225** (2010), no. 3, 1634–1668.

[DJ91] M.W. Davis and T. Januszkiewicz, *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. J. **62** (1991), no. 2, 417–451.

[GAP15] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.7.7*, 2015.

[GLdM13] S. Gitler and S. López de Medrano, *Intersections of quadrics, moment-angle manifolds and connected sums*, Geom. Topol. **17** (2013), no. 3, 1497–1534.

[GS03] A. Garrison and R. Scott, *Small covers of the dodecahedron and the 120-cell*, Proc. Amer. Math. Soc. **131** (2003), no. 3, 963–971.

[HLM92] H.M. Hilden, M.T. Lozano, and J.M. Montesinos, *On the Borromean orbifolds: geometry and arithmetic*, Topology '90 (Columbus, OH, 1990), Ohio State Univ. Math. Res. Inst. Publ., vol. 1, de Gruyter, Berlin, 1992, pp. 133–167.

[HLMW87] H.M. Hilden, M.T. Lozano, J.M. Montesinos, and W.C. Whitten, *On universal groups and three-manifolds*, Invent. Math. **87** (1987), no. 3, 441–456.

[LdM89] S. López de Medrano, *Topology of the intersection of quadrics in* $\mathbf{R}^n$, Algebraic topology (Arcata, CA, 1986), Lecture Notes in Math., vol. 1370, Springer, Berlin, 1989, pp. 280–292.

[LdM14] ———, *Singularities of homogeneous quadratic mappings*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM **108** (2014), no. 1, 95–112.

[Löb31] F. Löbell, *Beispiele geschlossener dreidimensionaler Clifford-Kleinscher räume negativer krümmung*, Berichte Leipzig **83** (1931), 167–174.

[Mon13] J.M. Montesinos, *On integral quadratic forms having commensurable groups of automorphisms*, Hiroshima Math. J. **43** (2013), no. 3, 371–411.

[S$^+$15] W.A. Stein et al., *Sage Mathematics Software (Version 6.6)*, The Sage Development Team, 2015, `http://www.sagemath.org`.

[Thu80] W.P. Thurston, *The geometry and topology of three-manifolds*, Princeton University, available at `http://library.msri.org/nonmsri/gt3m`, 1980.

[Ves98] A.Yu. Vesnin, *Volumes of Löbell* 3*-manifolds*, Mat. Zametki **64** (1998), no. 1, 17–23.

[Ves10] ———, *Volumes and normalized volumes of right-angled hyperbolic polyhedra*, Atti Semin. Mat. Fis. Univ. Modena Reggio Emilia **57** (2010), 159–169 (2011).

[Wal80] C.T.C. Wall, *Stability, pencils and polytopes*, Bull. London Math. Soc. **12** (1980), no. 6, 401–421.