BRAID MONODROMY AND TOPOLOGY OF PLANE CURVES

ENRIQUE ARTAL BARTOLO

Groupes de Tresses, Invariants Quantiques
et Géométrie des Variétés de Dimension 3
Luminy, June 6-10 2005

Contents

Introduction

The main goal of braid monodromy is to study the topology of algebraic curves, both affine and projective. This is very useful in studying the topology of projective complex surfaces, which was the primary aim of O. Zariski in the foundational paper [Zar29]. This study was also the starting point of braid monodromy by B. Moishezon [Moi81, Moi83, Moi85].

Let $C \subset \mathbb{C}^2$ be an affine curve defined as the zero locus of a non-zero reduced polynomial $f(x, y) \in \mathbb{C}[x, y]$ of degree d. We will suppose also that C has no vertical asymptote (including vertical components), i.e., f is monic as polynomial in y. Let $d_f(x) \in \mathbb{C}[x]$ be the y-discriminant of $f(x, y)$. Let $W := \{t \in \mathbb{C} \mid d_f(t) \neq 0\}$. A vertical line $L_t : \{x = t\}$ intersects C at exactly d points if and only if $t \in W$.

If $\bar{C} \subset \mathbb{P}^2$ is a reduced projective plane curve, we obtain the affine situation by considering a point and a line L through P; for generic choices of P, L and coordinates such that $P = [0 : 1 : 0]$ and L has equation $z = 0$. We identify $\mathbb{C}^2$ with $\mathbb{P}^2 \setminus L$ via $(x, y) \mapsto [x : y : 1]$ and we obtain in this way an affine curve $C := \bar{C} \cap \mathbb{C}^2$.

Braid monodromy is intuitively obtained as follows. Let $\star \in W$ and consider a closed path α in W based at $\star$; the intersections of the vertical lines over this path with C define in a natural way a geometric braid starting and ending in $\mathbf{y}_\star := \{y \in \mathbb{C} \mid f(\star, y) = 0\}$.

The ideas of Zariski and van Kampen are the key point of the definition of braid monodromy. Zariski gave a method to compute the fundamental group G of the complement of an affine or projective curve which was completed by van Kampen. The first point is that if one takes a generic line L, then the fundamental group of $L \setminus C$ generates G. Zariski showed how to construct *monodromy* relations using the above *motions* and van Kampen proved that they were a complete system of relations.

We can give the following short *history* of this invariant:

- Implicitely involved in Zariski-van Kampen for the computation of the fundamental group of the complement of an affine (or projective) plane curve.
- Explicitely stated by Chisini [Chi33] in the thirties.
- Intensively applied by Moishezon (and Teicher) [MT88, MT91, MT94a, MT94b, MT96] in the eighties. Braid monodromy of some curves is characterized. There are other contributions by Libgober [Lib86, Lib89], Salvetti [Sal88], Cohen-Suciu [CS97].
- There are also works concerning the relationship between topology and braid monodromy, with special attention to invariants characterizing braid monodromies: Kulikov-Teicher [KT00], Kharlamov-Kulikov [KK01, KK02] and the Artal-Carmona-Cogolludo *et al.* [Car03, ACCT01, ACC03, ACC05, ACCM03].
- There is a closed relationship with symplectic geometry, see the works of Auroux [Aur00] among others.

1. Preliminaries

Consider the polynomial f as a mapping $f : \mathbb{C} \to V$, where

$$V := \{h(y) \in \mathbb{C}[y] \mid h \text{ monic}, \ \deg(h) = d\}.$$

Considering the roots of polynomials, V is naturally identified with the quotient of $\mathbb{C}^d$ by the coordinate-permutation action of Σ_d, the symmetric group in d symbols. The image of the big diagonal in $\mathbb{C}^d$ in V is the discriminant variety

$$D := \{h(y) \in V \mid h \text{ has multiple roots}\}.$$

Let us denote $X := V \setminus D$, and let us identify subsets of $\mathbb{C}$ (with cardinality d) with polynomials in V. It is well-known that $\pi_1(X; \mathbf{y}_\star)$ is naturally isomorphic with the braid group $\mathbb{B}_{\mathbf{y}_\star}$ of homotopy classes of geometric braids based at $\mathbf{y}_\star$.

Let us recall that a geometric braid is a set $\boldsymbol{\gamma}$ of d paths from $[0,1]$ to $\mathbb{C}$ such that at each $t \in [0,1]$, $\boldsymbol{\gamma}(t) \in X$. A braid is closed and based at some $\mathbf{y} \in X$ if $\boldsymbol{\gamma}(0) = \boldsymbol{\gamma}(1) = \mathbf{y}$. The fundamental groupoid of X is identified with the braid groupoid, where if $\mathbf{y}_1, \mathbf{y}_2 \in X$, $\mathbb{B}_{\mathbf{y}_1,\mathbf{y}_2}$ is the set of homotopy classes (relative to $\{0,1\}$) starting at $\mathbf{y}_1$ and ending at $\mathbf{y}_2$; the grupoid product is defined by yuxtaposition and the inverse by reversing orientation.

Let us denote $\mathbf{y}_0 := \{-1, \ldots, -d\}$; the standard Artin braid group $\mathbb{B}_d$ will be identified with $\mathbb{B}_{\mathbf{y}_0}$. We recall the well-known Artin presentation:

$$\langle \sigma_1, \ldots, \sigma_{d-1} \mid [\sigma_i, \sigma_j] = 1, \quad j > i+1, \quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \rangle . \tag{1.1}$$

The standard generator σ_i, $i = 1, \ldots, d-1$ is identified with the following geometric braid. We consider constant paths at j, $j \neq i, i+1$ and we consider also the two semicercles of radii $\frac{1}{2}$, centered at $-\frac{2i+1}{2}$, with extremities $i, i+1$ and counterclockwise parametrized.

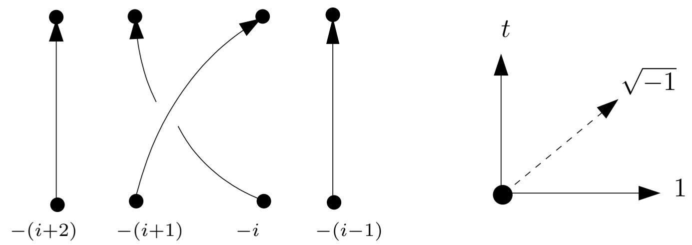

FIGURE 1. Standard braid σ_i

We consider in $\mathbb{C}$ a lexicographic order:

$$x \prec y \Longleftrightarrow \begin{cases} \Re x < \Re y & \text{or} \\ \Re x = \Re y \text{ and } \Im x < \Im y. \end{cases}$$

Given any $\mathbf{y} := \{y_1, \dots, y_d\} \in X$, we can order it such that $y_1 \succ \dots \succ y_d$. It is easily seen that the segments starting at $-i$ and ending at y_i, $1 \leq i \leq d$, define a braid $\sigma(\mathbf{y}_0, \mathbf{y})$. These braids allow us to identify $\mathbb{B}(\mathbf{y}_1, \mathbf{y}_2)$ with $\mathbb{B}_d$.

Remark 1.1. Any other choice of braids from $\mathbf{y}_0$ to $\mathbf{y}$, for any $\mathbf{y} \in X$, provide suitable identifications $\mathbb{B}(\mathbf{y}_1, \mathbf{y}_2) \sim \mathbb{B}_d$, $\mathbf{y}_1, \mathbf{y}_2 \in X$.

2. First definitions of braid monodromy

The restriction $f_| : W \to X$ induces a mapping $f_* : \pi_1(W; \star) \to \mathbb{B}_{\mathbf{y}_\star}$; this mapping is the first notion of *braid monodromy*. Using either the braid $\sigma(\mathbf{y}_0, \mathbf{y}_\star)$ or any other braid in $\mathbb{B}(\mathbf{y}_0, \mathbf{y}_\star)$, we can conjugate the former f_* and we obtain a representation $\nabla : \pi_1(W; \star) \to \mathbb{B}_d$, which will be also called braid monodromy. This mapping is only well-defined up to conjugation.

It is well-known that the fundamental group of a punctured plane is a free group with rank the number of punctures. We have encountered two punctured planes: $\mathbb{C} \setminus \mathbf{y}_\star \equiv L_\star \setminus C$ and W. Let us recall that $\mathbb{B}_{\mathbf{y}_\star}$ has a natural right action on $\pi_1(\mathbb{C} \setminus \mathbf{y}_\star; \blacktriangle)$ (denoted exponentially) as far as $|\blacktriangle|$ is big enough.

This notion allows us to give a proof of the Zariski-van Kampen theorem.

Theorem 2.1 (Zariski [Zar29], van Kampen [Kam33])**.** *Let $K := \pi_1(L_\star \setminus C; (\star, \blacktriangle))$ and let us consider the natural action of $\mathbb{B}_{\mathbf{y}_\star}$ on K. Then, the morphism $K \to \pi_1(\mathbb{C}^2 \setminus C; (\star, \blacktriangle))$ induced by the injection is surjective and its kernel is the normal subgroup generated by*

$$\left\{ \mu^{-1} \cdot \mu^{f_*(\gamma)} \mid \mu \in K, \gamma \in \pi_1(W; \star) \right\}.$$

Proof. Let us denote C^ψ the union of C and the vertical lines L_t such that $d_f(t) = 0$. The restriction $\pi : \mathbb{C}^2 \setminus C^\psi \to W$ of the first projection is a locally trivial fiber bundle. Since the universal covering of the base is contractible, the homotopy long exact sequence becomes a short exact sequence:

$$1 \to K \to \pi_1(\mathbb{C}^2 \setminus C^\psi; (\star, \blacktriangle)) \to \pi_1(W; \star) \to 1. \tag{2.1}$$

In order to study it, we look for a suitable model. Let us consider the disk Δ_x of radius $| * |$; it has been chosen big enough in order to contain all the punctures of W in its interior. The disk

Δ_y of radius $|\blacktriangle|$ is also big enough in order to contain all the punctures at the fiber $L_\star$ and to ensure that

$$C \cap (\Delta_x \times \partial\Delta_y) = \emptyset. \tag{2.2}$$

The restriction $\pi : (\Delta_x \times \Delta_y) \setminus C^\psi \to \Delta_x \cap W$ is a strong deformation retract of the former π.

From (2.1) it is possible to provide a presentation of $\pi_1(\mathbb{C}^2 \setminus C^\psi; (\star, \blacktriangle))$. The condition (2.2) allows to consider the splitting $j_* : \pi_1(W; \star) \to \pi_1(\mathbb{C}^2 \setminus C^\psi; (\star, \blacktriangle))$ defined by the inclusion $j : \Delta_x \cap W \hookrightarrow (\Delta_x \times \Delta_y) \setminus C^\psi$, $j(x) := (x, \blacktriangle)$.

The definition of the natural action of $\mathbb{B}_{\mathbf{y}_\star}$ on K implies that if $\alpha \in \pi_1(W; \star)$ then we have the following equalities in $\pi_1(\mathbb{C}^2 \setminus C^\psi; (\star, \blacktriangle))$:

$$(j_*(\alpha))^{-1} \cdot \mu \cdot j_*(\alpha) = \mu^{f_*(\alpha)}, \quad \forall \mu \in K. \tag{2.3}$$

Since the extremities of the sequence (2.1) are free groups, using (2.3) we determine a presentation of the middle term.

Using transversality arguments, we have that the inclusion induces an epimorphism $\pi_1(\mathbb{C}^2 \setminus C^\psi; (\star, \blacktriangle)) \twoheadrightarrow \pi_1(\mathbb{C}^2 \setminus C; (\star, \blacktriangle))$. It is trivial that $j_*(\pi_1(W; \star))$ is in the kernel of this mapping; the famous van Kampen theorem was produced in order to check that it normally generates the kernel. Transversality arguments help also in the proof of this result, which gives the theorem. □

Remark 2.2. In his foundational paper [Zar29], Zariski uses *motions* in pencil of lines which give relations. He computes how these relations behave locally and braid monodromy helps to handle with them both locally and globally. If we consider a projective curve $\bar{C}$ with the generic conditions, it is enough to replace $\pi_1(L_\star \setminus C)$ by $\pi_1(\bar{L}_\star \setminus \bar{C})$.

3. Final definition of braid monodromy

In order to present braid monodromy in a more manageable way, let us define a special class of bases for $\pi_1(W; \star)$. We recall first the definition of meridians.

Definition 3.1. Let Z be a connected projective manifold and let H be a hypersurface of Z. Let $\star \in Z \setminus H$ and let K be an irreducible component of H. A homotopy class $\gamma \in \pi_1(Z \setminus H; \star)$ is called a *meridian about* K *with respect to* H if γ has a representative δ satisfying the following properties:

(a) there is a smooth complex analytic disk $\Delta \subset Z$ transverse to H such that $\Delta \cap H = \{\star'\} \subset K$ (transversality implies that $\star'$ is a smooth point of H).
(b) there is a path α in $Z \setminus H$ from $\star$ to $\star'' \in \partial\Delta$.
(c) $\delta = \alpha \cdot \beta \cdot \alpha^{-1}$, where β is the closed path obtained by traveling from $\star''$ along $\partial\Delta$ in the positive direction.

Let us suppose that all the punctures are contained in the interior of the closed disk Δ_x of radius $|\star|$. Let $\mathbb{C} \setminus W := \{x_1, \ldots, x_r\}$. There exists a basis $\gamma_1, \ldots, \gamma_r$ of $\pi_1(W; \star)$ such that each γ_i is a meridian of x_i and such that $\gamma_r \cdot \ldots \cdot \gamma_1$ is homotopic to the boundary of Δ_x (parametrized counterclockwisely). Note that, in particular, $(\gamma_r \cdot \ldots \cdot \gamma_1)^{-1}$ is a meridian of the point at infinity.

Definition 3.2. A basis $\gamma_1, \ldots, \gamma_r$ of $\pi_1(W; \star)$ is *geometric* if each γ_i is a meridian of x_i and $\gamma_r \cdot \ldots \cdot \gamma_1$ is homotopic to the boundary of Δ_x (parametrized counterclockwisely). If $(\gamma_r \cdot \ldots \cdot \gamma_1)^{-1}$ is only a meridian of the point at infinity, we say that the basis is *pseudogeometric*.

Remark 3.3. The canonical example of geometric basis of $\pi_1(\mathbb{C}\setminus \mathbf{y}_0;\star)$, for some $\star \gg 0$, is $\gamma_1,\dots,\gamma_d$, obtained as follows:

- γ_i is a meridian of $-i$.
- γ_i runs along the real axis avoiding counterclockwisely the points $-j$, $j<i$.

The natural action of $\mathbb{B}_d$ on this free group is given by:

$$\gamma_i^{\sigma_j} := \begin{cases} \gamma_i & i\neq j, j+1,\\ \gamma_{j+1} & i=j,\\ \gamma_{j+1}\cdot\gamma_j\cdot\gamma_{j+1}^{-1} & i=j+1. \end{cases}$$

We can give the third notion of braid monodromy: it is $(\nabla(\gamma_1),\dots,\nabla(\gamma_r))\in(\mathbb{B}_d)^r$, where $\gamma_1,\dots,\gamma_r$ is a geometric basis. Since two meridians are always conjugate and ∇ is only defined up to conjugation, we may accept pseudogeometric basis.

Let us recall the notion of Hurwitz moves. Given a group G, $\mathbb{B}_r$ acts on G^r as follows. The action of the i^{th} standard generator of $\mathbb{B}_r$ (the i^{th} elementary Hurwitz move) is:

$$(g_1,\dots,g_r)\mapsto(g_1,\dots,g_{i-1},g_{i+1},g_{i+1}g_ig_{i+1}^{-1},g_{i+2},\dots,g_r).$$

Note that the product $g_r\cdot\ldots\cdot g_1$ remains invariant by Hurwitz moves. Note that this action commutes with the simultaneous conjugation in G and we have an action of $\mathbb{B}_r\times G$. The geometric interpretation of a well-known result of Artin is the following one.

Proposition 3.4 (Artin [Art47])**.** *The group $\mathbb{B}_r$ acts freely and transitively on the set of geometric bases of $\pi_1(W;\star)$.*

Definition 3.5. The *braid monodromy* of C is the orbit of $(\nabla(\gamma_1),\dots,\nabla(\gamma_r))\in(\mathbb{B}_d)^r$ by the action of $\mathbb{B}_r\times\mathbb{B}_d$, where $\gamma_1,\dots,\gamma_r$ is a pseudogeometric basis of $\pi_1(W;\star)$.

4. Puiseux monodromy and homotopy

In order to compute braid monodromy of a curve one needs to describe the braid associated with meridians in $\pi_1(W;\star)$. Let us fix such a meridian, which will have a decomposition $\delta=\alpha\cdot\beta\cdot\alpha^{-1}$ as in Definition 3.1. Recall that β is the boundary of a small disk such that in its center there is a vertical line L intersecting non-transversally the curve C. The braid associated to β is computed using the Puiseux expansion and will be positive in the standard generators.

Examples 4.1. Zariski [Zar29] was aware of this fact and he used it intensively. For example, with a suitable normalization, we can describe the braids associated to some non-transversality situations:

(a) An ordinary tangent (local equation $y^2-x=0$). The braid monodromy equals σ_1; the *non-trivial relation* is $\mu_1=\mu_2$.

(b) A generic node (local equation $y^2-x^2=0$). The braid monodromy equals σ_1^2; the *non-trivial relation* is $[\mu_1,\mu_2]=1$.

(c) An $\mathbb{A}_k$ singularity (local equation $y^2-x^{k+1}=0$). The braid monodromy equals σ_1^{k+1}; if $k=2$, the *non-trivial relation* is $\mu_1\cdot\mu_2\cdot\mu_1=\mu_2\cdot\mu_1\cdot\mu_2$.

(d) A flex of order k (local equation $y^{k+1}-x=0$). The braid monodromy equals $\sigma_k\cdot\ldots\cdot\sigma_1$; the *non-trivial relations* are $\mu_1=\cdots=\mu_k$. This fact was used in [Zar29] to prove that the fundamental group of the complement of a projective smooth curve is abelian.

We do not have an intepretation of the braid along α; note that it is not a closed braid but we can identify it with a closed braid using lexicographic braids. We will see how to compute the braids alon these α in some curves (strongly real curves); for general computations, one can use some computer programs (Carmona [Car03], `VKCurve` of D. Bessis and J. Michel).

Example 4.2 (Artal-Carmona-Cogolludo [ACC05])**.** Consider the projective curves $C_\beta := \{f_\beta g_\beta = 0\}$, $\beta^2 = 2$, where:

$$\begin{aligned} f_\beta(x,y,z) :=& y^2z^3 + (303 - 216\,\beta)\,yz^2x^2 + (-636 + 450\,\beta)\,yzx^3 + \\ &+ (-234\,\beta + 331)\,yx^4 + (-18\,\beta + 27)\,zx^4 + (18\,\beta - 26)\,x^5, \end{aligned}$$

and

$$g_\beta(x,y,z) := y + \left(\frac{10449}{196} - \frac{3645}{98}\,\beta\right) z + \left(-\frac{432}{7} + \frac{297}{7}\,\beta\right) x.$$

The component of degree 5 has singularities $\mathbb{E}_6, \mathbb{A}_3, \mathbb{A}_2$ and the line cuts it at two points with intersection multiplicities $1, 4$. We consider the affine curves obtained with $z = 1$ (the projection point is the $\mathbb{E}_6$ singular point and the line at infinity is its tangent line).

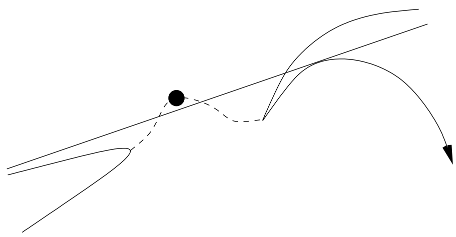

FIGURE 2. Real part of $C^{\text{aff}}_{\sqrt{2}}$

The braid monodromy for $\sqrt{2}$ equals: $[\sigma_2^8, \sigma_2^4 * \sigma_1^2, \sigma_2^3 * \sigma_1^3, \sigma_2 * \sigma_1^4, \sigma_1^{-3} * \sigma_2] \in \mathbb{B}_3^5$, where $a * b := aba^{-1}$.

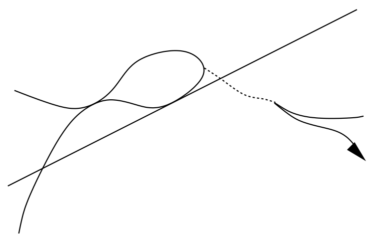

FIGURE 3. Real part of $C^{\text{aff}}_{-\sqrt{2}}$

The braid monodromy for $-\sqrt{2}$ equals: $[\sigma_2^3, (\sigma_2\sigma_1^{-1}\sigma_2) * \sigma_1, \sigma_2 * \sigma_1^8, \sigma_1^{-2} * \sigma_2^4, \sigma_1^{-3} * \sigma_2^2] \in \mathbb{B}_3^5$.

If we focus our attention on the decomposition of the braids as conjugation of *Puiseux braids*, we are a priori closer to the geometry. Let us consider a geometric basis of $\pi_1(W; \star)$.

Definition 4.3. A *Puiseux monodromy* of a curve C is an element of $(\mathbb{B}_d^2)^r$ obtained by considering the open braids starting at the base point and ending *near* the points in $\mathbb{C} \setminus W$ and the positive braids around the singular points obtained via Puiseux expansions. If we denote it as $((\alpha_i, \beta_i))_{i=1}^r$, we have:

- $(\alpha_i * \beta_i)_{i=1}^r$ represents braid monodromy.
- β_i is a positive braid.

Theorem 4.4 (Zariski-van Kampen(2)). *Let us suppose that $((\alpha_i, \beta_i))_{i=1}^r$ is a Puiseux monodromy of C. For the sake of simplicity we will suppose that each vertical line L_{x_i} contains only one point P_i of non transversality of C. Let r_i be the number of strings involved in β_i (say, the first ones): $r_i = (C \cdot L_{x_i})_{P_i}$. Then, if $\mu_1, \ldots, \mu_d$ is a geometric basis of K then $\pi_1(\mathbb{C}^2 \setminus C; (\star, \blacktriangle))$ has a presentation:*

$$\langle \mu_1, \ldots, \mu_d \mid (\mu_j^{-1} \mu_j^{\beta_i})^{\alpha_i^{-1}}, i = 1, \ldots, r, \quad j = 1, \ldots, r_i - 1 \rangle. \tag{4.1}$$

Proof. The relations obtained in Theorem 2.1 can be written as:

$$(\mu_j^{\alpha_i})^{-1} (\mu_j^{\alpha_i})^{\beta_i}, i = 1, \ldots, r, \quad j = 1, \ldots, d-1,$$

since the product of all relations for one line is a trivial relation. Note that $(\mu_j^{\alpha_i})_{j=1}^d$ is a geometric basis. If we would have chosen as generic fiber the base fiber of β_i, only the first $r_i - 1$ relations would be non trivial; this basis is related with the fixed one via α_i^{-1}. ☐

Theorem 4.5 (Libgober [Lib86]). *The complex associated with the presentation* (4.1) *has the homotopy type of $\mathbb{C}^2 \setminus C$. In particular, Puiseux monodromy determines this homotopy type.*

5. Braid monodromy and topology

Theorem 5.1 (Kulikov-Teicher [KT00],Carmona [Car03]). *Braid monodromy completely determines the topology type of the embedding of C.*

It is classically known that the composition of braid monodromy with the natural morphism $\mathbb{B}_d \twoheadrightarrow \Sigma_d$ determines the abstract topology. Moreover, if we follow Libgober's strategy in Theorem 4.5, one can find that Puiseux monodromy determines the topology type of the embedding of C. We consider a geometric basis of meridians giving a Puiseux monodromy $((\alpha_i, \beta_i))_{i=1}^r$. Over the disks bounded by β_i, we have a topological model of the curve, and we call this model with the braids α_i. We use product structure to extend the curve over $\mathbb{C}$.

Sketch of Kulikov-Teicher's proof. A priori, Puiseux monodromy could not be determined by braid monodromy. Let us suppose that (α, β) is a component of a braid monodromy. If $[\beta, \eta] = 1$, then we could replace (α, β) by $(\alpha\eta, \beta)$ and the corresponding braid monodromy would not change. Morevoer, this process allow to construct all possible Puiseux monodromies for a given braid monodromy.

Kulikov and Teicher use the presentation of the centralizer of a canonical generator σ_1 to show that the topological type does not change if we replace $(1_{\mathbb{B}_d}, \sigma_1)$ by $\eta, \sigma_1)$ for any generator of the centralizer of σ_1. The same method applies to powers of σ_1 and then, their proof works for curves with at most double points. It has been generalized by Manfredini-Pignatelli [MP00] to curves with singularities having only one Puiseux pair. ☐

Sketch of Carmona's proof. This proof works for arbitrary singularities. Braid monodromy determines the isotopy class of C outside the preimages of small disks around the points of $\mathbb{C} \setminus W$. Over these points, the topological type of the projection determines also the isotopy class. One must check that there is only one way to glue these isotopies and this is done using Waldhausen graph manifolds [Wal67a, Wal67b] (using Neumann plumbing calculus [Neu81]). ☐

It is not known if the converse is true. There are two partial converses which are true.

Theorem 5.2 (Carmona [Car03]). *The topological embedded of C with respect to a projection determines braid monodromy.*

Problem 5.3. *Find an effective way to* express *the topology in terms of braid monodromy.*

Given the affine curve C, we define the curve C^φ obtained as the projective closure of C^ψ and the line at infinity.

Theorem 5.4 (Artal-Carmona-Cogolludo [ACC03]). *The embedded topological type of C^φ (with C, the line at infinity and the point at infinity of vertical lines marked) determines the braid monodromy of C.*

Sketch of the proof. The main point is to consider the exact sequence (2.1). Since K is the subgroup generated by the meridians of K, even if the projection is not defined, (2.1) has an intrinsic meaning.

Since vertical lines are preserved, the notion of boundary of a big disk in a fiber is also intrinsic, and then, we can define geometric basis of the subgroup K. On the other side, the splitting j_* is also well-defined since, given a meridian in $\pi_1(W;\star)$ there is only one way to lift this meridian in a meridian of the corresponding vertical line and such that its conjugation action on K is braid-like.

Finally, the hypothesis allow also to give an intrinsic meaning to pseudogeometric basis of $\pi_1(W;\star)$. Since the natural action of $\mathbb{B}_d$ on the free group of rank d is effective, we can recover braid monodromy of a pseudogeometric basis of $\pi_1(W;\star)$. □

6. Projective curves and non-generic situations

Let $\bar{C}$ be a reduced projective plane curve. If L is a line in $\mathbb{P}^2$, we can identify $\mathbb{P}^2 \setminus L \equiv \mathbb{C}^2$ and $C := \bar{C} \setminus L$ becomes an affine curve. The choice of a point in L determine a pencil of (vertical) lines which allows us to construct a braid monodromy. If we choose *generically* this line, the obtained braid monodromy depends only on $\bar{C}$ and it is the *braid monodromy of C*.

The line can be chosen as follows. Consider:

- Tangent line at inflection points.
- Lines in the tangent cone of singularities.
- Lines joining two points which are either singular or inflections.
- Bitangent lines.
- Tangent lines through singular or inflection points.

We consider a point P outside this finite number of lines and a generic line L through this point.

Proposition 6.1. *Let $\bar{C}_0, \bar{C}_1 \subset \mathbb{P}^2$ be two curves of degree d such that there exists an equisingular continuous family $(C_t)_{t\in[0,1]}$ joining them. Then, they have the same braid monodromies.*

The original Zariski-van Kampen theorem applies to projective curves.

Theorem 6.2 (Zariski-van Kampen(3)). *Let $\bar{C} \subset \mathbb{P}^2$ be a projective curve of degree d and let $(\tau_1, \ldots, \tau_r) \in (\mathbb{F}_d)^r$ be a representative of its braid monodromy. Then:*

$$\pi_1(\mathbb{P}^2 \setminus \bar{C}) = \langle \mu_1, \ldots, \mu_d \mid \mu_i = \mu_i^{\tau_j}, \mu_d \cdot \ldots \cdot \mu_1 = 1 \rangle.$$

Remark 6.3.

(a) This is a consequence of Theorem 2.1 and classical van Kampen theorem.
(b) One can obtain less relations by taking a Puiseux monodromy as in Theorem 4.4.
(c) One can relax the genericity conditions: The result is true as far as the projection point is not in $\bar{C}$.
(d) If the projection point is in $\bar{C}$ and the line at infinity is its tangent cone, one can consider the braid monodromy of the resulting affine curve. With a careful analysis of braid monodromy and the topology of the affine curve at infinity, we can also obtain a presentation of the fundamental group of the complement of the projective curve.
(e) J. Carmona [Car03] has studied the additional information needed to handle braid monodromies for affine curves with asymptotes; D. Bessis has also similar results.

7. Braid monodromy and presentations of braid groups

There are two tools coming from representation theory that can help in the study of braid monodromy. The first one is a method by Libgober [Lib89]. If we compose braid monodromy with a representation of braid groups in a matrix group with coefficients in a principal ideal domain R (e.g. one-variable Laurent polynomial with coefficients in a field), using Fitting ideals we can associate to a braid monodromy an element in R. In the case of Burau representation one finds essentially the Alexander polynomial of the curve.

The other tool comes from finite representations. Let us suppose that we have computed two braid monodromies represented by $(\tau_1, \dots, \tau_r), (\eta_1, \dots, \eta_r) \in (\mathbb{B}_d)^r$.

(St-a) Check if there is a permutation of $(\eta_1, \dots, \eta_r)$ such that η_i is conjugated to τ_i. If it is not the case, they are not equivalent. If it is the case, we can perform Hurwitz moves in order and such that each η_i is conjugated to τ_i. Moreover we can reorder $(\tau_1, \dots, \tau_r)$ in order to have a decomposition of the r-uples in blocks containin the elements of the same conjugacy class. to group the different conjugacy classes in
(St-b) If (St-a) works, check if $\tau := \tau_r \cdot \dots \cdot \tau_1$ (*pseudo-Coxeter element* [Bri88]) and $\eta_r \cdot \dots \cdot \eta_1$ are conjugated. If it is the case, after a simultaneous conjugation we may suppose they are equal.
(St-c) If (St-b) works, let $H \subset \mathbb{B}_d$ be the centralizer of τ. Then, check if the subgroups generated by $(\tau_1, \dots, \tau_r)$ and $(\eta_1, \dots, \eta_r)$ are conjugated by an element in H (they are called *monodromy subgroups*). If it is the case, we can suppose that both groups and pseudo-Coxeter elements coincide. Let K be the normalizer of the monodromy subgroup in H.
(St-d) We can apply Zariski-van Kampen method to find the fundamental group of the complement of the curve.
(St-e) We can consider the orbit of $(\tau_1, \dots, \tau_r)$ by the action by conjugation by K and the action by Hurwitz moves respecting the blocks. Since these orbits are infinite, we can use a finite representation $\Psi : \mathbb{B}_d \to G$; the orbits are finite and the problem may be solved (e.g. in [GAP04]).

8. Applications

In this section we will treat two applications. Theorem 5.4 has proven the strength of braid monodromy in the detection of topological properties. We focus our attention on the so-called *algebraic Zariski pairs*. In order to see if their equivalent (and obtain a topological property) we can use several approaches:

Definition 8.1. [Art94] Two curves $C_1, C_2 \subset \mathbb{P}^2$ are a *Zariski pair* if there are regular neighbourhoods such that the pairs $(T(C_1), C_1)$ and $(T(C_2), C_2)$ are homeomorphic (combinatorial property) but $(\mathbb{P}^2, C_1)$ and $(\mathbb{P}^2, C_2)$ are not homeomorphic.

Combinatorics determine the first part since this property depends on degrees of irreducible components, topological types of singularities and relationship between local branches and global components. Examples of Zariski pairs are known from Zariski times. The first examples are distinguished in terms of Alexander polynomials. For other ones, we need a generalization of this invariant, Libgober's characteristic varieties [Lib01]. Some of them are distinguished by the fundamental group. In all these cases, some algebraic properties *explain* the differences between the members of the pairs.

Definition 8.2. An *algebraic Zariski pair* is a Zariski pair such that its members have conjugate equations in some number field.

Note that algebraic Zariski pairs cannot be distinguished by algebraic invariants. We have studied the curves of Examples 4.2.

We have computed non-generic braid monodromies. Though they are not priori so important as generic ones, they have some interesting properties:

- They help also to compute fundamental groups (with some care if the projection point is in the curve).
- Computations are easier since in general both d and r decrease.
- Non genericity can carry important properties of the curves and we can also obtain invariants. This fact is also strengthened because of Theorem 5.4 (see line arrangement example in [ACCM03]).

In Example 4.2, we have obtained different braid monodromy using a representation (e) of $\mathbb{B}_3$ in $\mathrm{GL}(2; \mathbb{Z}/32\mathbb{Z})$. In this cases the fundamental group of the complements of both projective curves is isomorphic to $\mathbb{Z} \times \mathrm{SL}(2; \mathbb{F}_7)$, see [ACC05]. As a consequence, if we add the non-transversal lines through the base points, we obtain an algebraic Zariski pair. Using this technique, we have also found algebraic Zariski pairs of arrangement of lines; in particular, we have found real arrangements with the same combinatorics and such that their complexifications are not homeomorphic [ACCM03]; in this example, we get the result using (St-c).

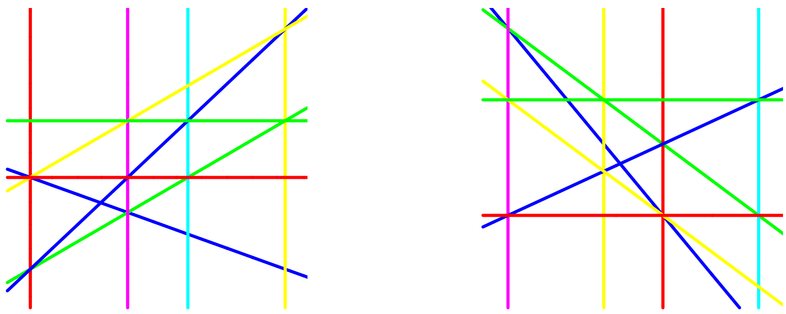

FIGURE 4. Algebraic Zariski pair of real arrangements

Braid monodromy is also useful in singularity theory, see the work of Brieskorn [Bri88] or the recent Ph.D thesis of Lönne. We present one application and we begin with some preliminaries which can be found in [Mil68] or [AGZV88].

Let us consider a germ of isolated singularity $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$. The main invariants of this germ are the monodromy of the Milnor fibration and the Seifert form of its link. By Milnor theory, there are $0 < \delta \ll \varepsilon \ll 1$ such that f has a representative

$$\bar{B}^4_\varepsilon(0) \cap f^{-1}(\bar{B}^2_\delta(0)) =: Y \xrightarrow{f} T := \bar{B}^2_\delta(0).$$

The algebraic link $K := \partial Y \cap f^{-1}(0) \subset \partial Y \cong S^3$ does not depend on the choice of (small enough) ε. Outside the origin f is a locally trivial fibration; its fiber (called Milnor fiber) F is homotopic to a wedge of μ S^1's (μ is called the Milnor number). This fibration is controlled by the monodromy acting on F. Extending suitably, F is a Seifert surface of K and its Seifert form can be used to recover the homological monodromy.

If we take a generic linear form ℓ and a small constant a, $f_a := f + a\ell$ becomes a morsification of f, i.e, it has only ordinary double points with different values; its number is μ. Picard-Lefschetz theory of vanishing cycles allows to find a base for the homology of F; moreover, the generic fiber of f_a is isotopic to the one of F and this isotopy extends to the fibrations over ∂T.

Let us consider $f_a : Y_a \to T_a$ (Y_a and T_a are isotopic to Y and T respectively). Let $t_1, \dots, t_\mu$ be the critical values of f_a (in the interior of T_a). Let us fix $\star \in \partial T_a$; we identify F with $f_a^{-1}(\star)$. Let us consider a geometric basis $\gamma_1, \dots, \gamma_\mu$ of meridians for $\pi_1(T_a \setminus \{t_1, \dots, t_\mu\}; \star)$. Let $\star_i$ be the intersection point of the circle and the arc of γ_i. Let $P_1, \dots, P_\mu$ be the critical points of f_a, and consider the Picard-Lefschetz vanishing cycle associated to P_i in the fiber over $\star_i$; transporting this cycle through the arc in γ_i we obtain a vanishing cycle c_i in F; $c_1, \dots, c_\mu$ provide a base of the homology of F.

Theorem 8.3. *The Seifert form in this basis is upper triangular. In particular, in this basis, the matrices for Seifert form, intersection form and monodromy determine each other.*

We are going to explain how braid monodromy helps us to compute the intersection form. These results are part of the Ph.D. thesis of M. Escario. Let us consider the polar map $\Phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that $\Phi(x, y) := (f(x, y), \ell(x, y))$.

After a change of coordinates we may suppose $\ell(x, y) = x$; the jacobian locus of this mapping is the zero locus of $\frac{\partial f}{\partial y}$ and its image Δ (the discriminant) is the zero locus of the $d_f(x, t)$ which is the discriminant of $f(x, y) - t$ with respect to y. If we replace f by $f + ax$, the new discriminant Δ_a is the zero locus of $d_f(x, t - ax)$.

The mapping Φ defines a finite ramified covering of the Milnor fiber of f onto a small disk in a vertical line; the ramification is concentrated on the intersection with Δ.

It is easily seen that the unique vertical line tangent to Δ is the vertical axis. The morsification is related with vertical lines after the change of variables $t \mapsto t - ax$; singular points of the morsification are relate with new vertical tangents.

The key point is to consider the braid monodromy of Δ associated to the projection $(t, x) \mapsto x$. There is a notion of *vanishing path* on the vertical lines near the vertical tangents and braid monodromy of Δ allows us to translate them to vanishing paths in a fixed vertical line. Using the ramified covering Φ we obtain the vanishing cycles in the Milnor fiber.

This method applies also to the so-called tame polynomials, introduced by Broughton. For these polynomials, adding a generic linear form produces a morsification. This method provides effective computations useful for the understanding of the topology of polynomials.

It would be interesting to work only with the discriminant of f in the following way. The local braid monodromy of Δ is easily computable. Since 0 is the only critical value of the projection, it is defined by a braid $\tau \in \mathbb{B}_k$ which correspond to a singularity having the vertical direction in the tangent cone. If we consider now the braid monodromy of Δ_a we have the following singular values:

- μ values corresponding to the μ vertical lines at smooth points.
- If Δ is not smooth, 0 is another singular value, which correspond to a singular point in general position.

We can choose a geometric basis for the punctured disk such that the μ first generators correspond the tangent to vertical lines and the last one (if it exists) to the singular point. Let $\alpha_1, \dots, \alpha_\mu, \beta$ be the associate braids ($\beta = 1_{\mathbb{B}_k}$, if Δ is smooth). Note that α_i is conjugated to σ_1. Note that

$$\tau = \beta \cdot \sigma_\mu \cdot \ldots \cdot \sigma_1. \tag{8.1}$$

Problem 8.4. *Let us suppose that we have two decompositions of τ such that the first braid is conjugated to β and the other μ braids are conjugated to σ_1. Are they in the same orbit by Hurwitz moves and conjugation by an element in the centralizer of β?*

Since the topological type of Δ is known, a positive solution to this problem would simplify the computation and allow the application of this method in good-at-infinity polynomials, which is a family larger than the one of tame polynomials.

References

[ACC03] E. Artal, J. Carmona, and J.I. Cogolludo, *Braid monodromy and topology of plane curves*, Duke Math. J. **118** (2003), no. 2, 261–278.

[ACC05] ———, *Effective invariants of braid monodromy*, Trans. Amer. Math. Soc. (2005), aceptado.

[ACCM03] E. Artal, J. Carmona, J.I. Cogolludo, and M. Marco, *Topology and combinatorics of real line arrangements*, Compositio math. (2003), aceptado.

[ACCT01] E. Artal, J. Carmona, J.I. Cogolludo, and H. Tokunaga, *Sextics with singular points in special position*, J. Knot Theory Ramifications **10** (2001), no. 4, 547–578.

[AGZV88] V.I. Arnol′d, S.M. Guseĭn-Zade, and A.N. Varčenko, *Singularities of differentiable maps. Vol. II*, Birkhäuser Boston Inc., Boston, MA, 1988.

[Art47] E. Artin, *Theory of braids*, Ann. of Math. (2) **48** (1947), 101–126.

[Art94] E. Artal, *Sur les couples de Zariski*, J. Algebraic Geom. **3** (1994), no. 2, 223–247.

[Aur00] D. Auroux, *Symplectic 4-manifolds as branched coverings of* $\mathbf{CP}^2$, Invent. Math. **139** (2000), no. 3, 551–602.

[Bri88] E. Brieskorn, *Automorphic sets and braids and singularities*, Braids (Santa Cruz, CA, 1986), Amer. Math. Soc., Providence, RI, 1988, pp. 45–115.

[Car03] J. Carmona, *Monodromía de trenzas de curvas algebraicas planas*, Ph.D. thesis, Universidad de Zaragoza, 2003.

[Chi33] O. Chisini, *Una suggestiva rappresentazione reale per le curve algebriche piane*, Ist. Lombardo, Rend., II. Ser. **66** (1933), 1141–1155.

[CS97] D.C. Cohen and A.I. Suciu, *The braid monodromy of plane algebraic curves and hyperplane arrangements*, Comment. Math. Helv. **72** (1997), no. 2, 285–315.

[GAP04] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4*, 2004, available at `(http://www.gap-system.org)`.

[Kam33] E.R. van Kampen, *On the fundamental group of an algebraic curve*, Amer. J. Math. **55** (1933), 255–260.

[KK01] V. Kharlamov and V.S Kulikov, *Diffeomorphisms, isotopies, and braid monodromy factorizations of plane cuspidal curves*, C. R. Acad. Sci. Paris Sér. I Math. **333** (2001), no. 9, 855–859.

[KK02] ______, *Deformation inequivalent complex conjugated complex structures and applications*, Turkish J. Math. **26** (2002), no. 1, 1–25.

[KT00] V.S. Kulikov and M. Teicher, *Braid monodromy factorizations and diffeomorphism types*, Izv. Ross. Akad. Nauk Ser. Mat. **64** (2000), no. 2, 89–120.

[Lib86] A. Libgober, *On the homotopy type of the complement to plane algebraic curves*, J. Reine Angew. Math. **367** (1986), 103–114.

[Lib89] ______, *Invariants of plane algebraic curves via representations of the braid groups*, Invent. Math. **95** (1989), no. 1, 25–30.

[Lib01] ______, *Characteristic varieties of algebraic curves*, Applications of algebraic geometry to coding theory, physics and computation (Eilat, 2001), Kluwer Acad. Publ., Dordrecht, 2001, pp. 215–254.

[Mil68] J. Milnor, *Singular points of complex hypersurfaces*, Annals of Mathematics Studies, no. 61, Princeton University Press, Princeton NJ, 1968.

[Moi81] B. Moishezon, *Stable branch curves and braid monodromies*, L.N.M. 862, Algebraic geometry (Chicago, Ill., 1980), Springer, Berlin, 1981, pp. 107–192.

[Moi83] ______, *Algebraic surfaces and the arithmetic of braids. I*, Arithmetic and geometry, Vol. II, Progr. Math., vol. 36, Birkhäuser Boston, Boston, MA, 1983, pp. 199–269.

[Moi85] ______, *Algebraic surfaces and the arithmetic of braids. II*, Combinatorial methods in topology and algebraic geometry (Rochester, N.Y., 1982), Contemp. Math., vol. 44, Amer. Math. Soc., Providence, RI, 1985, pp. 311–344.

[MP00] S. Manfredini and R. Pignatelli, *Generic covers branched over* $\{x^n = y^m\}$, Topology Appl. **103** (2000), no. 1, 1–31.

[MT88] B. Moishezon and M. Teicher, *Braid group technique in complex geometry. I. Line arrangements in* $\mathbf{C}\mathrm{P}^2$, Braids (Santa Cruz, CA, 1986), Amer. Math. Soc., Providence, RI, 1988, pp. 425–555.

[MT91] ______, *Braid group technique in complex geometry. II. From arrangements of lines and conics to cuspidal curves*, Algebraic geometry (Chicago, IL, 1989), Springer, Berlin, 1991, pp. 131–180.

[MT94a] ______, *Braid group techniques in complex geometry. III. Projective degeneration of* V_3, Classification of algebraic varieties (L'Aquila, 1992), Amer. Math. Soc., Providence, RI, 1994, pp. 313–332.

[MT94b] ______, *Braid group techniques in complex geometry. IV. Braid monodromy of the branch curve* S_3 *of* $V_3 \to \mathbf{C}\mathrm{P}^2$ *and application to* $\pi_1(\mathbf{C}\mathrm{P}^2 - S_3, *)$, Classification of algebraic varieties (L'Aquila, 1992), Amer. Math. Soc., Providence, RI, 1994, pp. 333–358.

[MT96] ______, *Braid group technique in complex geometry. V. The fundamental group of a complement of a branch curve of a Veronese generic projection*, Comm. Anal. Geom. **4** (1996), no. 1-2, 1–120.

[Neu81] W.D. Neumann, *A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves*, Trans. Amer. Math. Soc. **268** (1981), no. 2, 299–344.

[Sal88] M. Salvetti, *Arrangements of lines and monodromy of plane curves*, Compositio Math. **68** (1988), no. 1, 103–122.

[Wal67a] F. Waldhausen, *Eine klasse von* 3*-dimensionalen mannigfaltigkeiten I*, Invent. Math. **3** (1967), 308–333.

[Wal67b] ______, *Eine klasse von* 3*-dimensionalen mannigfaltigkeiten II*, Invent. Math. **4** (1967), 87–117.

[Zar29] O. Zariski, *On the problem of existence of algebraic functions of two variables possessing a given branch curve*, Amer. J. Math. **51** (1929), 305–328.

Departmento de Matemáticas, Universidad de Zaragoza Campus Plaza de San Francisco s/n, E-50009 Zaragoza SPAIN

E-mail address: artal@unizar.es