COMPLEX ORBIFOLDS, QUOTIENT SINGULARITIES, WEIGHTED PROJECTIVE PLANES AND SYMMETRIC CURVES

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1. Settings

(St1) Unless otherwise stated we will work with complex analytic varieties and holomorphic mappings.

(St2) Vectors in $\mathbb{R}^n, \mathbb{C}^n, \mathbb{Z}^n$, etc, are represented as column matrices or vectors.

(St3) Open sets of analytic varieties are analytic varieties. By abuse of language we may call analytic varieties to subsets of analytic varieties with non-empty interior and piecewise differentiable boundary, e.g. closed disks in $\mathbb{C}$, closed polydisks or closed balls in $\mathbb{C}^n$. Holomorphic mappings will stand for the restriction of holomorphic functions defined on an open superset.

(St4) The (closed) unit disk centered at the origin will be denoted by $\mathbb{D} \subset \mathbb{C}$; for radius r, it will be denoted by $\mathbb{D}_r$ and if the center is $w \in \mathbb{C}$, it will be $\mathbb{D}_r(w)$. The same conventions will be used for (open) balls $\mathbb{B}$ in $\mathbb{C}^n$.

Partially supported by MCIN/AEI/10.13039/501100011033 (grant code: PID2020-114750GB-C31) and by Departamento de Ciencia, Universidad y Sociedad del Conocimiento del Gobierno de Aragón (grant code: E22_20R: "Álgebra y Geometría").

(St5) Group actions will be left actions unless otherwise stated. If the group is abelian, we may mix right and left.

(St6) The cyclic group with additive notation is $\mathbb{Z}$ and with multiplicative notation it will be a free group with one generator. A free group with r generators will be denoted by $\mathbb{F}_r$.

(St7) Finite cyclic groups will be identified either with $\mathbb{Z}/p$ (additive notation) or with μ_p (multiplicative notation), the group of p-roots of unity in $\mathbb{C}$.

2. Complex orbifolds

Orbifolds are objects which locally model finite actions. The focus is not on the quotient but on the action.

Example 2.1. A well-known properly discontinuous action is the action by translation of a lattice Λ on $\mathbb{C}$; a lattice is an abelian subgroup of $(\mathbb{C},+)$ which generates $\mathbb{C}$ as an $\mathbb{R}$-vector space. The action is free and its quotient $\mathbb{T}_\Lambda := \mathbb{C}/\Lambda$ is a compact analytic curve of genus 1, an elliptic curve. The quotient map $\pi : \mathbb{C} \to \mathbb{T}_\Lambda$ is a cover, namely the universal cover of $\mathbb{T}_\Lambda$.

Example 2.2. Another well-known action is the action of μ_p on $\mathbb{C}$ by multiplication. This action is of course properly discontinuous and faithful but it is not free. Let us recall that the quotient $\mathbb{C}/\mu_p$ is identified with $\mathbb{C}$ via $[z] \equiv z^p$, but with this identification we are losing the information about the non-freeness of the action.

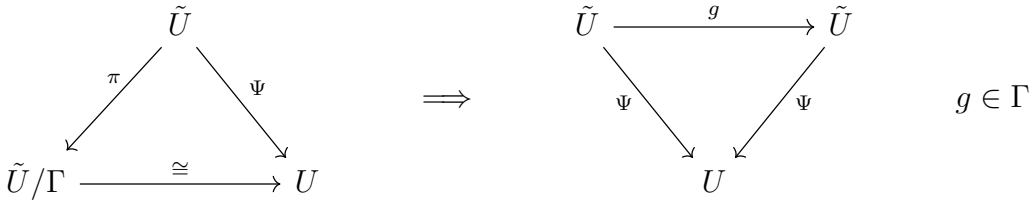
Definition 2.3. An *n-complex orbifold* X is the data of a Hausdorff second countable topological space $|X|$ with a maximal atlas of orbifold charts. An *orbifold chart* is a tuple $(\tilde{U}, U, \Gamma, \Psi)$ where

(O1) $\tilde{U}$ is an open subset of $\mathbb{C}^n$;

(O2) U is an open subset of $|X|$.

(O3) Γ is a finite group of analytic isomorphisms of $\tilde{U}$.

(O4) $\Psi : \tilde{U} \to U$ is a surjective continuous map such that these diagrams commute.



The targets of these charts form an open cover of $|X|$. Given two charts

$$(\tilde{U}_i, U_i, \Gamma_i, \Psi_i), \qquad i = 1, 2,$$

and $P \in U_1 \cap U_2$, there exists an orbifold chart $(\tilde{U}_P, U_P, \Gamma_P, \Psi_P)$, $U_P \subset U_1 \cap U_2$, $\#\Psi_P^{-1}(P) = 1$, and complex embeddings $\Phi_i : \tilde{U}_P \to \tilde{U}_i$ fitting in the following commutative diagram:

$$\begin{array}{ccccc} \tilde{U}_1 & \xleftarrow{\Phi_1} & \tilde{U}_P & \xrightarrow{\Phi_2} & \tilde{U}_2 \\ {\scriptstyle\Psi_1}\downarrow & & \downarrow{\scriptstyle\Psi_P} & & \downarrow{\scriptstyle\Psi_2} \\ U_1 & \longleftarrow & U_P & \longhookrightarrow & U_2 \end{array} \tag{2.1}$$

Moreover, the maps Φ_i are (Γ_i, Γ_P)-equivariants, inducing monomorphisms $\gamma_i : \Gamma_P \to \Gamma_i$, such that if $g \in \Gamma_P$ and $Q \in \tilde{U}_P$ then $\Phi_P(g \cdot Q) = \gamma_i(g) \cdot \Phi_i(g)$.

Remark 2.4. The map of an orbifold chart is not bijective, and this is why the definition of the change of charts is more involved. Note that if $g \in \Gamma_i$, then $g \cdot \Phi_i$ can replace Φ_i in the above diagram.

Examples 2.5.

(1) Analytic varieties are special cases of orbifolds.
(2) Let X be an analytic manifold and let G be a group acting on X faithfully and properly discontinuously. Then, there is a natural orbifold structure X_G such that $|X_G| = X/G$.
(3) A orbifold chart $(\tilde{U}, U, \Gamma, \Psi)$ of an orbifold X defines an orbifold $\tilde{U}_\Gamma$ such that $\left|\tilde{U}_\Gamma\right| = U$; this orbifold is an open suborbifold of X.
(4) For the action of lattices on $\mathbb{C}$, since the action is free we can safely identify $\mathbb{T}_\Lambda$ with $\mathbb{C}_\Lambda$.
(5) The orbifold $\mathbb{C}_{(m)} := \mathbb{C}_{\mu_m}$ keeps the information of the action and $\left|\mathbb{C}_{(m)}\right| \equiv \mathbb{C}$. We denote anagously $\mathbb{D}_{(m)}$.

The intermediate charts are special for the points P and they desserve to be named.

Definition 2.6. Let X be an orbifold and let $P \in |X|$. We say that a chart $(\tilde{U}, U, \Gamma, \Psi)$ is P-special if $\#\Psi^{-1}(P) = 1$. Note that if $\tilde{P}$ is this preimage, then $g \cdot \tilde{P} = \tilde{P}$, $\forall g \in G$. The isomorphism type of Γ does not depend on the chart and $\Gamma =: \mathrm{Iso}_X^P$ is called the *isotropy group* of P. A point P is called *orbifold-regular* (resp. *orbifold-singular*) in X if Iso_X^P is (resp. not) trivial.

Proposition 2.7. *Let X be an orbifold. There is a maximal stratification $\mathscr{S}$ by relatively closed subsets, such that the isotropy groups remain constant in each stratum and for each chart the preimages are Zariski-open sets of analytic subvarieties.*

Definition 2.8. An *orbifold map* $f : X \to Y$, where X is an n-orbifold and Y is an m-orbifold is the data of:

(OM1) a continuous map $|f| : |X| \to |Y|$;

(OM2) for each pair of orbifold charts $(\tilde{U}_X, U_X, \Gamma_X, \Psi_X)$ of X and $(\tilde{U}_Y, U_Y, \Gamma_Y, \Psi_Y)$ such that $|f|(U_X) \subset U_Y$, a holomorphic lifting $\tilde{f}_{\Psi_Y,\Psi_X}$ satisfying

$$\begin{array}{ccc} \tilde{U}_X & \xrightarrow{\tilde{f}_{\Psi_Y,\Psi_X}} & \tilde{U}_Y \\ {\scriptstyle \Psi_X}\downarrow & & \downarrow{\scriptstyle \Psi_Y} \\ U_X & \xrightarrow[f]{} & U_Y, \end{array}$$

and for any $\gamma_X \in \Gamma_X$ there exists $\gamma_Y \in \Gamma_Y$, such that $\tilde{f}_{\Psi_Y,\Psi_X} \circ \gamma_X = \gamma_Y \circ \tilde{f}_{\Psi_Y,\Psi_X}$.

(OM3) The liftings are compatible with (2.1).

Note that the liftings are only meaningful for the points with non-trivial isotropy.

Remark 2.9. With the same ideas we can define continuous or differentiable orbifolds, and continuous and differentiable orbifold maps.

Example 2.10. There is a unique orbifold map $F : \mathbb{C}_{(m)} \to \mathbb{C}$ such that $|F| : \mathbb{C}/\mu_m \to \mathbb{C}$ is given by $|F|([z]) = z^m$. The lifting defining F is the identity (or the multiplication by $\zeta \in \mu_m$). With the identification $\mathbb{C}/\mu_m \equiv \mathbb{C}$, this map is actually the identity. Note that there is no orbifold map $G : \mathbb{C} \to \mathbb{C}_{(m)}$ such that $|G| = |F|^{-1}$, since $|F|^{-1}(w) = [\sqrt[m]{w}]$ which is well-defined and continuous but it cannot be lifted to a holomorphic map.

3. Orbifold fundamental group

Remark 3.1. With the same ideas we can define continuous or differentiable orbifolds, and continuous and differentiable orbifold maps.

Definition 3.2. Let X be an orbifold.

(i) An *orbifold path* is an orbifold continuous map $\gamma : [0,1] \to X$.

(ii) An *orbifold loop* is an orbifold map whose end points coincide.

(iii) An *orbifold homotopy relative to the end points* between two orbifold paths $\gamma_0, \gamma_1 : [0,1] \to X$ is an orbifold continuous map $H : [0,1] \times [0,1] \to X$ such that the restriction of H is constant in $\{j\} \times [0,1]$, and γ_j in $[0,1] \times \{j\}$, $j = 0, 1$.

Example 3.3. Let $X = \mathbb{D}_{(2)}$ and let $\gamma, \delta : [0,1] \to X$ be two orbifold paths, such that $|\gamma| = |\delta|$, see Figure 1.

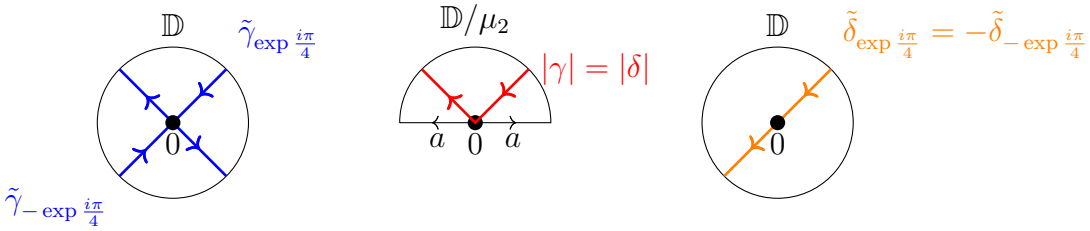


Figure 1. Paths in $\mathbb{D}_{(2)}$ and their liftings.

Definition 3.4. Let X be an orbifold, and let $x \in |X|$ be a regular point. The *orbifold fundamental group* $\pi_1^{\mathrm{orb}}(X; x)$ *of* X *at* x, is the group of orbifold homotopy classes (relative to the end points) of orbifold loops based at x.

This definition mimics the classical one, as it happens with its properties.

Properties 3.5.

($\pi_1^{\mathrm{orb}}1$) If X is a manifold, then it coincides with the standard fundamental group.

($\pi_1^{\mathrm{orb}}2$) It is functorial for orbifold maps up to orbifold homotopy.

($\pi_1^{\mathrm{orb}}3$) If X is a manifold and G acts faithfully and properly discontinuously, then the following exact sequence holds:

$$1 \longrightarrow \pi_1(X) \longrightarrow \pi_1^{\mathrm{orb}}(X_G) \longrightarrow G \longrightarrow 1.$$

In particular, if X is simply connected, $\pi_1^{\mathrm{orb}}(X_G) \cong G$.

One important tool for the computation of fundamental groups is the Seifert-van Kampen Theorem. It is also valid for the orbifold fundamental group.

Theorem 3.6 (Seifert-van Kampen Theorem)**.** *Let X be an orbifold obtained by the union of two open orbifolds U, V such that $U \cap V$ decomposes in orbifold connected components $\{W_i\}_{i \in I}$. Fix one index $0 \in I$. We make the following choices:*

(a) *A point $x_i \in W_i$ for $i \in I$;*
(b) *paths $\gamma_{U,i} : [0,1] \to U$, $\gamma_{V,i} : [0,1] \to V$ from x_0 to x_i for $i \in I \setminus \{0\}$; we denote $\delta_i := [\gamma_{V,i} \cdot \gamma_{U,i}^-]$.*

Then,

$$\pi_1^{\mathrm{orb}}(X; x_0) \cong \frac{\pi_1^{\mathrm{orb}}(U; x_0) * \pi_1^{\mathrm{orb}}(V; x_0) * \left\langle \underset{i \in I \setminus \{0\}}{\gamma_i} \middle| - \right\rangle}{R}$$

where R is the normal subgroup generated by

$$\{(j_U)_*(g) \cdot (j_V)_*(g)^{-1} \mid g \in \pi_1^{\mathrm{orb}}(W_0; x_0)\}$$

and

$$\{\delta_i \cdot (j_{i,U})_*(g) \cdot \delta_i^{-1} \cdot (j_{i,V})_*(g)^{-1} \mid g \in \pi_1^{\mathrm{orb}}(W_i; x_i)\}, \quad i \in I \setminus \{0\},$$

where

$$\begin{array}{ccc} \pi_1^{\mathrm{orb}}(W_i; x_i) & \xrightarrow{(j_{i,U})_*} & \pi_1^{\mathrm{orb}}(U; x_0) \\ g & \longmapsto & \gamma_{U,i} \cdot (j_U)_*(g) \cdot \gamma_{U,i}^{-1} \end{array} \qquad \begin{array}{ccc} \pi_1^{\mathrm{orb}}(W_i; x_i) & \xrightarrow{(j_{i,V})_*} & \pi_1^{\mathrm{orb}}(V; x_0) \\ g & \longmapsto & \gamma_{V,i} \cdot (j_V)_*(g) \cdot \gamma_{V,i}^{-1} \end{array}$$

Examples 3.7. We combine Seifert-van Kampen Theorem with ($\pi_1^{\mathrm{orb}}3$) to compute some orbifold fundamental groups.

(1) $\pi_1^{\mathrm{orb}}(\mathbb{C}_{(m)}) \cong \pi_1^{\mathrm{orb}}(\mathbb{D}_{(m)})$; a generator is given by the orbifold loop $\gamma(t) := \left[\exp \frac{2i\pi t}{m}\right]$.

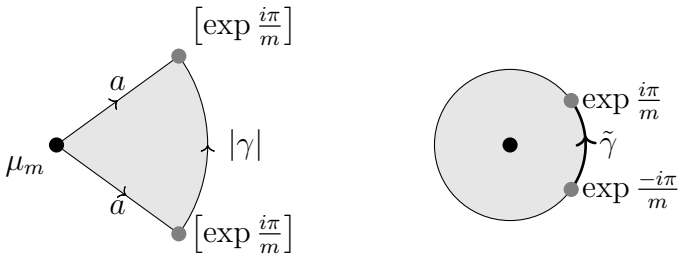


Figure 2. $\pi_1^{\mathrm{orb}}\left(\mathbb{C}_{(m)}; \left[\exp \frac{i\pi}{m}\right]\right) \cong \mu_m$.

(2) Let $X := \mathbb{P}^1_{(m_1,\ldots,m_r)}$ be an orbifold such that $\left|\mathbb{P}^1_{(m_1,\ldots,m_r)}\right| = \mathbb{P}^1$ and there are points $P_1, \ldots, P_r$ such that $\mathrm{Iso}_X^{P_i} \cong \mu_{m_i}$, $i = 1, \ldots, r$. The orbifold structure comes from the maximal atlas containing the following orbifold charts:

(b) If $P \neq P_i$, take a usual chart $\Psi_P : \mathring{\mathbb{D}} \to U_P$, such that $\Psi_P(0) = P$ and $P_1, \ldots, P_r \notin U_P$. The orbifold chart is $(\mathring{\mathbb{D}}, U_P, \{1\}, \Psi_P)$.

(b) If $P = P_i$, take a usual chart $\Psi_P : \mathring{\mathbb{D}} \to U_P$, such that $\Psi_P(0) = P_i$ and $P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_r \notin U_P$. Let $\tilde{\Psi}_P : \mathring{\mathbb{D}} \to U_P$ defined by $\tilde{\Psi}_P(z) = \Psi_P(z^{m_i})$ The orbifold chart is $(\mathring{\mathbb{D}}, U_P, \mu_{m_i}, \tilde{\Psi}_P)$.

Applying Seifert-van Kampen Theorem we obtain

$$\pi_1^{\mathrm{orb}}(\mathbb{P}^1_{(m_1,\ldots,m_r)}) = \langle \gamma_1, \ldots, \gamma_r | \gamma_1^{m_1} = \cdots = \gamma_r^{m_r} = \gamma_1 \cdot \ldots \cdot \gamma_r = 1 \rangle .$$

Analogously for $\mathbb{C}_{(m_1,\ldots,m_r)}$, its fundamental group is $\mu_{m_1} * \cdots * \mu_{m_r}$. The same ideas work for any Riemann surface.

(3) Let $G := \mu_{m_1} \times \mu_{m_2}$ acting on $\mathbb{C}^2$ as $(\zeta_1, \zeta_2) \cdot (x_1, x_2) := (\zeta_1 x_1, \zeta_2 x_2)$ and let $X := (\mathbb{C}^2)_G$ the induced orbifold. The map $(x_1, x_2) \mapsto (x_1^{m_1}, x_2^{m_2})$ identifies $\mathbb{C}^2/G$ and $\mathbb{C}^2$. Under this identification the stratification is

$$\underset{\mathrm{Iso}=\{1\}}{(\mathbb{C}^*)^2} \sqcup \underset{\mathrm{Iso}=\mu_{m_2}}{\mathbb{C}^* \times \{0\}} \sqcup \underset{\mathrm{Iso}=\mu_{m_1}}{\{0\} \times \mathbb{C}^*} \sqcup \underset{\mathrm{Iso}=\mu_{m_1} \times \mu_{m_2}}{\{(0,0)\}}$$

The fundamental group is G.

(4) Let

$$G := \frac{\mu_m^3}{\{(\zeta, \zeta, \zeta) \mid \zeta \in \mu_m\}} \cong \mu_m^2$$

acting on $\mathbb{P}^2$ as $[(\zeta_0, \zeta_1, \zeta_2)] \cdot [x_0 : x_1 : x_2] := [\zeta_0 x_0 : \zeta_1 x_1 : \zeta_2 x_2]$ and let $X := (\mathbb{P}^2)_G$ the induced orbifold. The Kummer map $[x_0 : x_1 : x_2] \mapsto [x_0^m : x_1^m : x_2^m]$ identifies $\mathbb{P}^2/G$ and $\mathbb{P}^2$. Under this identification the regular stratum is the complement of the axes in $\mathbb{P}^2$. The axes without the vertices isotropy groups μ_m, respectively, and G is the isotropy group of the vertices. The fundamental group is G and X is called a *Kummer orbifold*.

We are going to define two important orbifolds (local and global) to be used later.

Example 3.8. Let $p_x, p_y \in \mathbb{Z}_{>0}$ be coprime integers; let us denote $\omega := (p_x, p_y)$. We consider the following action of $\mathbb{C}^*$ on $\mathbb{C}^2 \setminus \{0\}$:

$$t \cdot (x, y) := (t^{p_x} x, t^{p_y} y).$$

The closure of the orbits have equations $\alpha x^{p_y} = \beta y^{p_x}$ for some $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{0\}$. Let us denote by X the quotient and by $[x : y]_\omega$ the equivalence class of (x, y). Following the description of the projective line we can consider the following *charts*

if we denote by U_x the image in X of $x \neq 0$ and by U_y the image in X of $y \neq 0$ (open sets in the quotient topology):

$$\begin{array}{ccc} \mathbb{C}_x \xrightarrow{\Psi_x} U_x & & \mathbb{C}_y \xrightarrow{\Psi_y} U_y \\ y \longmapsto [1:y]_\omega, & & x \longmapsto [x:1]_\omega. \end{array}$$

These maps are continuous and surjective but they are not injective. If $\zeta \in \mu_{p_x}$, then $\Psi_x(\zeta^{p_y}t) = [1:\zeta^{p_y}t]_\omega = [\zeta^{p_x}:\zeta^{p_y}t]_\omega = [1:t]_\omega$; since p_y is coprime with p_x it is not difficult to check that $\Psi_x(t_1) = \Psi_x(t_2)$ if and only if $t_2 = \zeta t_1$ for some $\zeta \in \mu_{p_x}$. We can see $(\mathbb{C}_x, U_x, \mu_{p_x}, \Psi_x)$ as an orbifold chart, and something similar happens with $(\mathbb{C}_y, U_y, \mu_{p_y}, \Psi_y)$; moreover, taking suitable domains where roots are holomorphic, the charts are compatible and they define an orbifold structure on X which will be called weighted projective line of weights $\omega = (p_x, p_y)$ and denoted by $\mathbb{P}^1_\omega = \mathbb{P}^1_{(p_x,p_y)}$; actually the notation is consistent with the one of the Example 3.7(2). The map

$$\begin{array}{ccc} \mathbb{P}^1_\omega & \longrightarrow & \mathbb{P}^1 \\ [x:y]_\omega & \longmapsto & [x^{p_y}:y^{p_x}] \end{array} \tag{3.1}$$

is a bijective orbifold map, but its inverse is not an orbifold map. Note that $\mathbb{P}^1_\omega$ is simply connected.

Example 3.9. There are some generalizations of this map, namely, the so-called *forget map* $\mathbb{P}^1_{(m_1,\dots,m_r)} \to \mathbb{P}^1$, given by the identity, is an orbifold map but its inverse is not. If n_i is a divisor of n_i the identity induces also a *partial forget map* $\mathbb{P}^1_{(m_1,\dots,m_r)} \to \mathbb{P}^1_{(n_1,\dots,n_r)}$.

4. Orbifold covers

In the classical case, fundamental groups are strongly related to covers. The same thing happens for the orbifold fundamental group.

Definition 4.1. An orbifold map $p : Y \to X$ is an *orbifold cover* if for all $x \in |X|$ there exists an x-special orbifold chart $(\tilde{U}_x, U_x, \Gamma_x, \Psi_x)$ such that

$$|p|^{-1}(U_x) = \bigsqcup_{y \in |p|^{-1}(x)} V_y$$

and there exist for each $y \in |p|^{-1}(x)$ a y-special orbifold chart $(\tilde{V}_y, V_y, \Gamma_y, \Psi_y)$ such that the lifting $\tilde{p}_y$ is biholomorphic:

$$\begin{array}{ccc} \tilde{V}_y & \xrightarrow[\cong]{\tilde{p}_y} & \tilde{U}_x \\ {\scriptstyle \Psi_y}\downarrow & & \downarrow{\scriptstyle \Psi_x} \\ V_y & \xrightarrow[|p|]{} & U_x. \end{array}$$

Example 4.2. If X is a manifold and G is a group acting faithfully and properly discontinuously, then the quotient map induces an orbifold map $X \to X_G$.

As a particular example, if $(\tilde{U}, U, \Gamma, \Psi)$ is an orbifold chart then Ψ induces an orbifold cover from $\tilde{U}$ to $\tilde{U}_\Gamma$.

Basically, all the usual cover theory can be transposed to orbifold covers:

(OC1) Orbifold paths and homotopies can be lifted.

(OC2) The map induced by an orbifold covers on orbifold fundamental groups is injective. As in the classical case, we will usually identify the orbifold of the orbifold fundamental group of the domain with a subgroup of the orbifold fundamental group of the base via the above monomorphism.

(OC3) The notion of Galois (or regular) orbifold cover also makes sense. As in the classical case it is equivalent to the group of the domain to be a normal subgroup of the group of the cover, and its quotient is the group of diffeomorphisms of the domain commuting with the cover.

(OC4) The notion of monodromy also works, and the orbifold covers of an orbifold are parametrized (up to isomorphism) by the conjugacy class of subgroups of the orbifold group. In particular, universal orbifold covers always exist and are simply connected.

(OC5) As a practical case if X is an orbifold, $x \in |X|$ is a regular point and $\rho : \pi_1^{\mathrm{orb}}(X; x) \to G$ is an epimorphism, then there is, up to isomorphism, a unique cover $p_\rho : X_\rho \to X$ such that for any $\hat{x} \in X_\rho$, $\pi_1^{\mathrm{orb}}(X_\rho; \hat{x}) = \ker \rho$.

As in the classical case, if $p : Y \to X$ is a G-Galois orbifold cover and Y is a manifold, then X is isomorphic to Y_G. In order to extend this property to the case when Y is an orbifold, let us define group actions on orbifolds.

Definition 4.3. Let X be an orbifold and let G be a group. A *(left) orbifold action* of G on X is a homomorphism of groups from G to the group of orbifold diffeomorphisms of X. If φ_g is the image of $g \in G$ and $x \in |X|$, we denote the action by $g \cdot x := \varphi_g(x)$. The action is properly discontinuous if the induced action on $|X|$ is.

Given a faithful and properly discontinuous action of a group G on an orbifold X, we are going to define the orbifold X_G, determined by this action. The space $|X_G|$ equals $|X|/G$. Let us define the orbifold charts. Let $x \in |X|$. There are two groups directly related to x, namely its isotropy group Iso_X^x and the isotropy group $G_x := \{g \in G \mid g \cdot x = x\}$ of the action of G on $|X|$; both are finite.

Let us associate a third group. Fix an x-special orbifold chart $(\tilde{U}_x, U_x, \mathrm{Iso}_X^x, \Psi_x)$. Since G_x is finite we can restrict this chart to a smaller one such that $\varphi_g(U_x) = U_x$, $\forall g \in G_x$. From the action definition we can construct liftings.

$$\begin{array}{ccc} \tilde{U}_x & \xrightarrow{\tilde{\varphi}_g} & \tilde{U}_x \\ {\scriptstyle \Psi_x}\downarrow & & \downarrow{\scriptstyle \Psi_x} \\ U_x & \xrightarrow[\varphi_g]{} & U_x \end{array}$$

We define $\Gamma_{G,x}$ as the group of liftings $\tilde{\varphi}_g$ of φ_g, for $g \in G_x$; it is the central term of the following exact sequence:

$$1 \longrightarrow \Gamma_x \longrightarrow \Gamma_{G,x} \longrightarrow G_x \longrightarrow 1.$$

Let us denote $\Phi_x : \tilde{U}_z \to U_x/G_x$ be the composition of Ψ_x and the quotient map. Then, $(\tilde{U}_x, U_x/G_x, \Gamma_{G,x}, \Phi_x)$ is an $[x]$-special orbifold chart. All these charts are compatible and determine the orbifold X_G. In particular, $\mathrm{Iso}_{X_G}^{[x]} = \Gamma_{G,x}$.

5. Pencils of curves and orbifolds

Let us interpret a famous example of Zariski (completed by Oka and others) in terms of orbifolds. Let $f_2, f_3 \in \mathbb{C}[x, y, z]$ be *generic* homogeneous polynomials in three variables of degrees 2 and 3, respectively. Let $f_6 := f_3^2 - f_2^2$; its zero locus in $\mathbb{P}^2$ is an irreducible sextic curve C_6 with six cuspidal points, which are in the conic C_2 defined by f_2; we denote C_3 the cubic defined by f_3.

Set $\omega := (2,3)$. Let $B := C_2 \cap C_3$, $Y := \mathbb{P}^2 \setminus B$, and $X := \mathbb{P}^2 \setminus C_6$. Under the genericity assumption, B is a finite set with six elements, and Y is the maximal subset in $\mathbb{P}^2$ where the following orbifold map can be defined:

$$\begin{aligned} Y &\xrightarrow{F} \mathbb{P}^1_\omega \\ [x:y:z] &\longmapsto [f_2(x,y,z) : f_3(x,y,z)]_\omega. \end{aligned}$$

Let us check that it is actually an orbifold map. Let $P \in C_2 \cap Y$, i.e., $F(P) = [0:1]_\omega$. We pick up a small chart $\Psi : \mathbb{D}_r \to Y$ chart of $\mathbb{P}^2$ containing P with local coordinates (u,v) such that $f_2(\Psi(u,v)) = u$; let $h(u,v) := f_3(\Psi(u,v))$. We choose the chart small enough such that h does not vanish. Then, the lifting looks like

$$\begin{array}{ccc} (u,v) & \xmapsto{\tilde{F}} & \dfrac{u}{h(u,v)^{\frac{2}{3}}} \\ \Big\downarrow \Psi & & \Big\downarrow \Psi_y \\ \Psi(u,v) & \xrightarrow{F} [u : h(u,v)]_\omega = \left[\dfrac{u}{h(u,v)^{\frac{2}{3}}} : 1\right]_\omega & \end{array}$$

Something similar happens for $P \in C_3$. Let $[a:b]_\omega \in \mathbb{P}^1_\omega$. Then:

$$\begin{aligned} F^{-1}([a:b]_\omega) &= \{[x:y:z] \in Y \mid [f_2(x,y,z) : f_3(x,y,z)]_\omega = [a:b]_\omega\} \\ &= \{[x:y:z] \in Y \mid \exists t \in \mathbb{C}^3 f_2(x,y,z) = t^2 a, f_3(x,y,z) = t^3 b\} \\ &= \{[x:y:z] \in Y \mid b^2 f_2(x,y,z)^3 = a^3 f_3(x,y,z)^2\}. \end{aligned}$$

Hence, $F^{-1}([1:1]_\omega) = C_6 \setminus B$ and for generic $[a:b]_\omega \in \mathbb{P}^1_\omega$, $F^{-1}([a:b]_\omega)$ is an irreducible sextic deprived of its singular points in B; in particular, it is path-connected.

Even though F is not a cover, (non unique) liftings of orbifold paths can be done. Let $P \in Y$ in a generic fiber of F, i.e., $F(P) := p$ is a regular point of the orbifold $\mathbb{P}^1_\omega$. Let γ be an orbifold loop in $Z := \mathbb{P}^1_\omega \setminus \{[1:1]_\omega\}$ based at p. It can be lifted to a path in Y starting at P; this path will end in $F^{-1}(p)$ and it can be extended to a loop $\tilde{\gamma}$ by a path contained in $F^{-1}(p)$. Then, $F \circ \tilde{\gamma}$ is orbifold homotopic to γ. He have seen that $F_{|*} : \pi_1(Y;P) \to \pi_1^{\mathrm{orb}}(Z;p)$ is a surjective morphism. Note that Z is isomorphic to $\mathbb{C}_{(2,3)}$, and then, $\pi_1(Y;P)$ admits $\mu_2 * \mu_3$ as a quotient and it is not abelian. It is well-known that $F_{|*}$ is an isomorphism

but non-trivial computations are needed to prove it. There is a particular case (a non-generic one) where this isomorphism can be proved in a relatively easy way. Since $\mu_2 * \mu_3$ is Hopfian, this fact implies isomorphism in the general case.

Remark 5.1. The above ideas can be expressed in a more simple way using the forget map of (3.1). In this case the map F is written as

$$\begin{aligned} Y &\xrightarrow{F} \mathbb{P}^1 \equiv \mathbb{P}^1_{(2,3)} \\ [x:y:z] &\longmapsto [f_2(x,y,z)^3 : f_3(x,y,z)^2], \end{aligned}$$

where $\mathrm{Iso}^{[1:0]}_{\mathbb{P}^1_{(2,3)}} \cong \mu_2$ and $\mathrm{Iso}^{[0:1]}_{\mathbb{P}^1_{(2,3)}} \cong \mu_3$, following the notation of Example 3.7(2).

Let us see how we can relate pencils and 1-orbifolds with more than two singular points. In the same work of Zariski, the fundamental group of the tricuspidal quartic is computed. Let us approach this computation using orbifolds.

One equation for the tricuspidal quartic C_4 is

$$f_4(x,y,z) := x^2y^2 + y^2z^2 + z^2x^2 - 2xyz(x+y+z) = 0,$$

and its bitangent is $C_1 :\ f_1(x,y,z) := x+y+z = 0$. Let us consider the following conic and cubic for ζ a primitive cubic root of unity:

$$\begin{aligned} C_2 :\ & f_2(x,y,z) := xy + \zeta^{-1}xz + \zeta yz = 0 \\ C_3 :\ & f_3(x,y,z) := x^2(y-z) + y^2(z-x) + z^2(x-y) + 3\left(\zeta - \zeta^{-1}\right)xyz = 0. \end{aligned}$$

We have $4f_2^3 + f_3^2 = f_4 f_1^2$. Let $B := C_2 \cap C_3$, a finite set with four points, the three cusps of C_4 and one tangency point in C_1. The map

$$\begin{aligned} \mathbb{P}^2 \setminus B &\xrightarrow{F} \mathbb{P}^1 \\ [x:y:z] &\longmapsto [f_2(x,y,z)^3 : f_3(x,y,z)^2], \end{aligned}$$

defines an orbifold map if we consider the target as $\mathbb{P}^1_{(2,3)}$, where the singular points of order $2,3$ are $[0:1], [1:0]$, respectively. Note that $F(C_4) = F(L) = [1:-4]$. The important point is that the restriction of F to $\mathbb{P}^2 \setminus C_4$ defines an orbifold map to $\mathbb{P}^1_{(2,3,2)}$, with generic connected fiber. In particular, $\pi_1(\mathbb{P}^2 \setminus C_4)$ has a surjection onto the triangle group $(2,2,3)$, i.e., the dihedral group of 6 elements.

Let us summarize these ideas.

Proposition 5.2. *Let $\mathcal{P}$ be a primitive pencil of curves of degree d in $\mathbb{P}^2$. Let $C_1, \dots, C_r$ and $D_1, \dots, D_r$ curves in $\mathbb{P}^2$ such that $C_i + m_i D_i$ is a curve in the pencil. Let $A_1, \dots, A_s$ other fibers of the pencil. Then,*

$$\pi_1\left(\mathbb{P}^2 \setminus \left(\bigcup_{j=1}^{s} A_j \cup \bigcup_{j=1}^{r} C_j\right)\right) \cong \left\langle \underset{j=1,\dots,s}{\gamma_j}, \underset{j=1,\dots,r}{\delta_j} \;\middle|\; \underset{j=1,\dots,r}{\delta_j^{m_j}} = 1, \prod_{j=1}^{s} \gamma_j \cdot \prod_{j=1}^{s} \delta_j = 1 \right\rangle.$$

Oka generalized Zariski's computations to arbitrary coprime p, q. An unpublished work of Cogolludo and Matei provides examples for any r.

6. Quotient singularities and orbifolds

Quotient singularities are normal singularities whose associated local ring is the ring of invariant actions of a regular local ring by the action of a finite group. It is not exactly the same notion of an orbifold but it is closely related. Orbifolds are more general since it is more than the quotient space, but the orbifold language also allows to treat these simple singularities in a simpler way.

Definition 6.1. Let G be a finite group acting (faithfully) on $\mathscr{O}_{\mathbb{C}^n,0}$. A *quotient singularity* is a germ (X, x) such that $\mathscr{O}_{X,x} = \mathscr{O}_{\mathbb{C}^n,0}^G$ the invariant functions by the action of G. The natural projection $\pi : (\mathbb{C}^n, 0) \to (X, x)$ induces also an orbifold structure, making (X, x) a *local quotient orbifold*.

If the singular locus has codimension at least 2 the two notions coincide. Even if in some cases we may be only interested in the singularity, the orbifold language may be useful.

Remark 6.2. All these actions are locally conjugated to linear actions.

Example 6.3. Let us consider the standard action of μ_p on $\mathbb{C}$ by multiplication. In this case $\mathscr{O}_{\mathbb{C},0} := \mathbb{C}\{x\}$ and $\mathscr{O}_{X,x} = \mathbb{C}\{x^p\}$ and hence $(X, x) \cong (\mathbb{C}, 0)$. The orbifold $\mathbb{C}_{\mu_p} = \mathbb{C}_{(p)}$ takes into account the action, $\mathrm{Iso}^{[0]}_{\mathbb{C}_{(p)}} \cong \mu_p$, although $\left|\mathbb{C}_{(p)}\right| \equiv \mathbb{C}$.

Example 6.4. Let us consider the diagonal action of μ_3 on $\mathbb{C}^2$. We have $\mathscr{O}_{\mathbb{C}^2,0} := \mathbb{C}\{x, y\}$ and $\mathscr{O}_{X,x} = \mathbb{C}\{x^3, x^2y, xy^2, y^3\}$. In this case $\mathbb{C}^2/\mu_3$ has an actual isolated singularity, the orbit of the origin, which is isomorphic to (X, x). In this case the orbifold $\mathbb{C}^2{}_{\mu_3}$ carries as much information as $\mathbb{C}^2/\mu_3$. There is an isomorphism

$$\mathbb{C}\{x^3, x^2y, xy^2, y^3\} \cong \frac{\mathbb{C}\{u, v, w, t\}}{\langle v^2 - uw, vw - ut, w^2 - vt\rangle},$$

and it is simpler to work with the *fake* coordinates x, y than with the *actual* coordinates u, v, w, t and their restrictions.

Let us give some notation for abelian quotient singularities and orbifolds.

Definition 6.5. Let $\mathbf{d} \in \mathbb{Z}^r_{>0}$, and $A \in \mathrm{Mat}(r \times n; \mathbb{Z})$. The *abelian orbifold* $(\mathbf{d}|A)$, associated to $\mathbf{d}$ and A, is the orbifold $\mathbb{C}^n_{\mu_{d_1} \times \cdots \times \mu_{d_r}}$ for the action

$$(\zeta_1, \ldots, \zeta_r) \cdot (x_1, \ldots, x_n) := \left(\prod_{i=1}^r \zeta_i^{a_{i1}} x_1, \ldots, \prod_{i=1}^r \zeta_i^{a_{n1}} x_n \right).$$

When $r = 1$, it will be represented as $\left(\frac{1}{d}; a_1, \ldots, a_n\right)$.

It is important to see that those actions need not to be faithful, and there are many pairs $(\mathbf{d}, A)$ representing the same action. Finding a *good representative* may involve cumbersome arithmetic computations.

Lemma 6.6. *Any* $(\mathbf{d}|A)$ *action as above on* $\mathbb{C}^n$ *can be represented with at most* n *rows. If the codimension of the singular locus is at least* 2*, then at most* $n-1$ *rows are enough.*

For $n = 1$, all the actions are of the type $\left(\frac{1}{d}; 1\right)$ and they correspond with the orbifold $\mathbb{C}_{(d)}$. Let us see some cases for $n = 2$.

Example 6.7. An abelian action such that the quotient is isomorphic to $\mathbb{C}^2$ is

$$\left(\begin{array}{c|cc} p & 1 & 0 \\ q & 0 & 1 \end{array}\right),$$

i.e., it corresponds to the orbifold $\mathbb{C}_{(p)} \times \mathbb{C}_{(q)}$. Its orbifold fundamental group is $\mu_{pq} \cong \mu_p \times \mu_q$.

Example 6.8. We are going to see local 2-orbifolds where only one point has non-trivial isotropy groups. They are of the form $(d|a\ b)$, where $\gcd(d, a) = \gcd(d, b) = 1$, actually it is enough to consider $(d|1\ b)$ with $\gcd(d, b) = 1$. For these orbifolds, their structure gives as much information as the variety does.

Proposition 6.9. *Let* (X, x) *be a* 2*-orbifold obtained as the quotient of an abelian group on* $(\mathbb{C}^2, 0)$*. Then,* $(|X|, x)$ *is of type* $(d|1\ b)$ *with* $\gcd(d, b) = 1$ *and for* $P = [(0, t)]$ *(resp.* $[(t, 0)]$*),* $t \neq 0$*,* $\mathrm{Iso}^P_X \cong \mu_p$ *(resp.* μ_q*), with* $p, q \geq 1$*. The group acting is isomorphic to* $\mu_{\mathrm{lcm}(p,q)d} \times \mu_{\gcd(p,q)}$*. If* $\gcd(p, q) = 1$*, then the action is* $\left(\frac{1}{dpq}; q, pb\right)$.

The actions where the singular points are of codimension 2 are called *small*; small abelian actions on $\mathbb{C}^2$ are always cyclic. For any orbifold $\left(\frac{1}{d}; a, b\right)$ we have a forget map onto a small orbifold, namely $\left(\frac{\gcd(d,a),\gcd d,b}{d}; \frac{a}{\gcd(a,d)}, \frac{b}{\gcd(b,d)}\right)$.

The orbifolds $\mathbb{P}^1_{(p,q,r)}$ for $(p,q,r) = (2,3,3), (2,3,4), (2,3,5)$, or $(2,2,n)$, $n \geq 2$, provide examples of non-abelian small actions. The general idea is as follows. Consider the universal orbifold cover which turns out to be $\pi_{(p,q,r)} : \mathbb{P}^1 \to \mathbb{P}^1_{(p,q,r)}$. The monodromy action is an action on $\mathbb{P}^1$ which can be lifted to $\mathbb{C}^2$ as a central extension of the finite group $\pi_1^{\text{orb}}(\mathbb{P}^1_{(p,q,r)})$.

Example 6.10. For $(2,2,n)$, we can consider the action generated by

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{\rho_\zeta} & \mathbb{C}^2 \\ (x,y) & \longmapsto & (\zeta x, \zeta^{-1} y) \end{array} \qquad \begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{\sigma} & \mathbb{C}^2 \\ (x,y) & \longmapsto & (y,x) \end{array}$$

where $\zeta \in \mu_n$. These diffeomorphisms generate the dihedral group D_{2n} of order $2n$ which has μ_n as the subgroup $\{\rho_\zeta \mid \zeta \in \mu_n\}$. Note that $\mathbb{C}^2{}_{\mu_n} = \left(\frac{1}{n}; 1, -1\right)$. The map

$$\begin{array}{ccc} \mathbb{C}^2 & \longrightarrow & \mathbb{C}^2 \\ (x,y) & \longmapsto & (x^n + y^n, xy) \end{array}$$

is a model for the quotient. Then $|\mathbb{C}^2{}_{D_{2n}}| \equiv \mathbb{C}^2$ with coordinates (u,v). The image of the diagonal is the curve $C_n : u^2 = 4v^n$. The isotropy stratification is given by

$$\underset{\text{Iso}=\{1\}}{(\mathbb{C}^2 \setminus C_n)} \sqcup \underset{\text{Iso}=\mu_2}{(C_n \setminus \{(0,0)\})} \sqcup \underset{\text{Iso}=D_{2n}}{\{(0,0)\}}$$

We can generalize this action to quotient orbifolds (X,x) such that $|X|$ is smooth.

Proposition 6.11. *The quotient orbifolds (X,x) such that $|X|$ is smooth are determined by the closure C of the 1-dimensional strata with non-trivial isotropy:*

1. *$C = \{x = 0\}$ with isotropy group μ_n.*
2. *$C = \{xy = 0\}$ with isotropy groups μ_n and μ_m in each irreducible component.*
3. *$C = \{y^2 - x^n\}$, with isotropy group μ_2.*
4. *$C = \{y^2 - x^3\}$, with isotropy group μ_n, for $n = 3, 4, 5$.*
5. *$C = \{y^2 - x^4\}$, with isotropy groups mu_2 and μ_n, for $n > 2$ in each irreducible component.*

(6) $C = \{y^2 - x^4\}$, *with isotropy groups* μ_n *and* μ_m, *for* $(n, m) = (3, 3), (3, 4)$, *and* $(3, 5)$ *in each irreducible component.*
(7) $C = \{y^2 - x^5\}$, *with isotropy group* μ_3.
(8) $C = \{y^2 - x^6\}$, *with isotropy groups* mu_n *and* μ_m, *for* $(n, m) = (2, 3), (2, 4)$, *and* $(2, 5)$ *in each irreducible component.*
(9) $C = \{y^2 - x^8\}$, *with isotropy groups* mu_2 *and* μ_3 *in each irreducible component.*
(10) $C = \{y^3 - x^3\}$, *with isotropy groups* mu_n, μ_m *and* μ_r, *for* spherical triples.
(11) $C = \{x(y^2 - x^p)\}$, $p > 2$, *with isotropy groups* μ_2 *for the branch or branches of* $y^2 = x^p$ *and* μ_n, $n \geq 2$ *for* $y = 0$.
(12) $C = \{x(y^2 - x^3)\}$, *with isotropy groups* μ_3 *for the cusp and* μ_2 *for* $y = 0$.
(13) $C = \{x(y^2 - x^4)\}$, *with isotropy groups* μ_a, μ_b *for branches of the tacnode and* μ_c *for* $y = 0$, *for* $(a, b, c) = (2, 3, 2), (2, 3, 3), (2, 4, 2), (2, 5, 2)$.
(14) $C = \{x(y^2 - x^6)\}$, *with isotropy groups* μ_2, μ_3 *for branches of the* $\mathbb{A}_5$ *and* μ_2 *for* $y = 0$.
(15) $C = \{y^3 - x^4\}$, *with isotropy group* μ_2.
(16) $C = \{y(y^2 - x^3)\}$, *with isotropy group* μ_a *for the cusp and* μ_b *for the smooth branch,* $(a, b) = (2, 2), (2, 3)$.
(17) $C = \{y^3 - x^5\}$, *with isotropy group* μ_2.

7. Weighted projective planes

Weighted projective lines appeared in Example 3.8. They are orbifold structures on the usual projective line. Its definition can be transposed to any dimension, where new varieties will appear. We restrict our attention to dimension 2.

Definition 7.1. Let $\omega := (p_x, p_y, p_z) \in \mathbb{Z}^3_{>0}$, $\gcd \omega = 1$. The *ω-weighted projective plane* $\mathbb{P}^2_\omega$ is the orbifold such that $|\mathbb{P}^2_\omega|$ is the quotient of $\mathbb{C}^3 \setminus \{0\}$ for the action of $\mathbb{C}^*$ defined by

$$t \cdot (x, y, z) := (t^{p_x} x, t^{p_y} y, t^{p_z} z)$$

where the orbifold charts are defined by the following maps. Let $[x : y : z]_\omega$ denote the class of (x, y, z) in the quotient, and let U_x (resp. U_y, U_z) be the open set of the quotient defined by the image of $x \neq 0$ (resp. $y \neq 0$, $z \neq 0$). The maps are:

$$\begin{array}{ccc} \mathbb{C}^2_x \xrightarrow{\Psi_x} U_x & \mathbb{C}^2_y \xrightarrow{\Psi_y} U_y & \mathbb{C}^2_z \xrightarrow{\Psi_z} U_z \\ (y, z) \longmapsto [1 : y : z]_\omega, & (x, z) \longmapsto [x : 1 : z]_\omega, & (y, z) \longmapsto [1 : y : z]_\omega. \end{array}$$

The orbifold charts defined by these maps are $(\mathbb{C}^2_a, U_a, \mu_{p_a}, \Psi_a)$ for $a \in \{x, y, z\}$.

Remark 7.2. Let us denote $P_x := [1 : 0 : 0]_\omega$, $P_y := [0 : 1 : 0]_\omega \mathrm{T}$, $P_z := [0 : 0 : 1]_\omega$, $L_x := \{x = 0\}$, $L_y := \{y = 0\}$, and $L_z := \{z = 0\}$. The isotropy groups for $P \in \mathbb{P}^2_\omega$ are:

- If $P \in U_x \cap U_y \cap U_z = \mathbb{P}^2_\omega \setminus (L_x \cup L_y \cup L_z)$, then $\mathrm{Iso}^P_{\mathbb{P}^2_\omega} = \{1\}$.
- If $P \in L_x \setminus \{P_y, P_z\} =: L^\vee_x$, then $\mathrm{Iso}^P_{\mathbb{P}^2_\omega} = \mu_{a_x}$, where $a_x := \gcd(p_y, p_z)$.
- If $P \in L_y \setminus \{P_x, P_z\} =: L^\vee_y$, then $\mathrm{Iso}^P_{\mathbb{P}^2_\omega} = \mu_{a_y}$, where $a_y := \gcd(p_x, p_z)$.
- If $P \in L_z \setminus \{P_x, P_y\} =: L^\vee_z$, then $\mathrm{Iso}^P_{\mathbb{P}^2_\omega} = \mu_{a_z}$, where $a_x := \gcd(p_x, p_y)$.
- If $P = P_a$, $a \in \{x, y, z\}$, then $\mathrm{Iso}^P_{\mathbb{P}^2_\omega} = \mu_{p_a}$.

If $\delta := (b_x, b_y, b_z)$, where $b_x := \frac{p_x}{a_y a_z}$, $b_y := \frac{p_y}{a_x a_z}$, and $b_z := \frac{p_z}{a_x a_y}$, then the map

$$\begin{aligned} \mathbb{P}^2_\omega &\longrightarrow \mathbb{P}^2_\delta \\ [x : y : z]_\omega &\longmapsto [x^{a_x} : y^{a_y} : z^{a_z}]_\delta \end{aligned}$$

is a bijective orbifold map (a forget map).

Lemma 7.3. $\pi^{\mathrm{orb}}_1(\mathbb{P}^2_\omega) = 1$.

Proof. As for the usual fundamental group it is not hard to see that the map $\pi^{\mathrm{orb}}_1(\mathbb{C}^2_{x\mu_{p_x}}) \to \pi^{\mathrm{orb}}_1(\mathbb{P}^2_\omega)$ induced by the inclusion on orbifold fundamental groups is surjective. In particular, $\pi^{\mathrm{orb}}_1(\mathbb{P}^2_\omega)$ is a quotient of μ_{p_x}; by the same reason, it is a quotient of μ_{p_y} and μ_{p_z} and the result follows. □

We will consider several types of orbifolds. One one side, if $G \subset \mathrm{Aut}\,\mathbb{P}^2_\omega$ is a finite subgroup, we will consider $(\mathbb{P}^2_\omega)_G$. On the other side, consider a stratification by relatively open strata such that the closure of each stratum is algebraic, and associate to each a compatible isotropy group. For this, sometimes we need to take out some 0-strata if it is not possible to associate an orbifold point.

Example 7.4. The group μ_n acts on $\mathbb{P}^2$ by

$$\zeta \cdot [x : y : z] := [\zeta^a x : \zeta^b y : z],$$

where $\gcd(a, b, n) = 1$, in order to ensure that the action is faithful. Let us describe the orbifold $S := S_{\mu_n,a,b} := {\mathbb{P}^2}_{\mu_n}$. The class of an element $[x : y : z]$ will be denoted as $[x : y : z]_{\mu_n,a,b}$.

If $P=[x:y:z]_{\mu_n,a,b}$, $xyz\neq 0$, then Iso_S^P is trivial. For $P=[x:y:0]_{\mu_n,a,b}$, with $xy\neq 0$ we have $\mathrm{Iso}_S^P\cong\mu_{\frac{n}{\gcd(n,a-b)}}$. In the same way for $P=[x:0:z]_{\mu_n,a,b}$, with $xz\neq 0$ we have $\mathrm{Iso}_S^P\cong\mu_{\frac{n}{\gcd(n,a)}}$ and for $P=[0:y:z]_{\mu_n,a,b}$, with $yz\neq 0$ we have $\mathrm{Iso}_S^P\cong\mu_{\frac{n}{\gcd(n,b)}}$. Finally,

$$(S,[1:0:0]_{\mu_n,a,b})\cong\left(\frac{1}{n};b-a,-a\right),$$
$$(S,[0:1:0]_{\mu_n,a,b})\cong\left(\frac{1}{n};a-b,-b\right),$$
$$(S,[0:0:1]_{\mu_n,a,b})\cong\left(\frac{1}{n};a,b\right).$$

8. Symmetric curves

Our main goal is to study symmetric curves using orbifolds. The idea is simple. Consider a curve $C\subset\mathbb{P}^2$ (it may work also in weighted projective planes), and assume that there is a finite subgroup $G\subset\operatorname{Aut}\mathbb{P}^2$ fixing C. Let $S:=\mathbb{P}^2{}_G$ and $\tilde{C}$ the image of C in S. One can obtain information of $(\mathbb{P}^2,C)$ using $(S,\tilde{C})$ since $\mathbb{P}^2\setminus C\to S\setminus\tilde{C}$ is an orbifold cover.

We are going to see in it in a particular but important example. We consider the icosidodecahedron arrangement. This can be understood as follows. Consider a regular dodecahedron $D_{12}\subset\mathbb{R}^3$ centered at the origin. Consider the 6 lines joining the center of D_{12} with each face and also the 10 lines joining the center of D_{12}. We can see these lines as points in $\mathbb{P}^2(\mathbb{R})$ and then lines in its dual and finally in the complexification. An affine real picture is shown in Figure 3 (one has to add the line at infinity). In the picture one sees immediately the action of the dihedral group of 10 elements, but the first description shows that the alternating group A_5 is a group of symmetries.

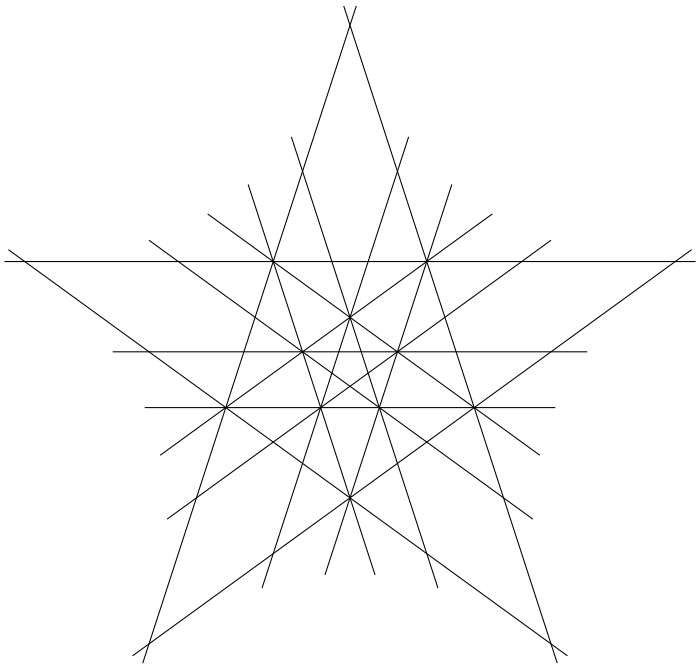


Figure 3. Affine picture of the icosidodecahedron arrangement

In order to apply the previous strategy, we need to understand this action. We fix a representation $\rho : A_5 \to \mathrm{PGL}(\mathbb{C}; 3)$ such that:

$$(1,\dots,5) \longmapsto \begin{bmatrix} -1 & 0 & 0 \\ 0 & -c_5 & \frac{1}{2}c_5 + \frac{3}{4} \\ 0 & -1 & -c_5 \end{bmatrix}$$

$$(1,2)(3,4) \longmapsto \begin{bmatrix} -\frac{4}{5}c_5 - \frac{1}{5} & \frac{4}{5}c_5 + \frac{1}{5} & c_5 \\ \frac{6}{5}c_5 + \frac{4}{5} & -\frac{1}{5}c_5 + \frac{7}{10} & -\frac{1}{4} \\ \frac{8}{5}c_5 + \frac{2}{5} & \frac{2}{5}c_5 - \frac{2}{5} & c_5 + \frac{1}{2} \end{bmatrix}$$

$$(5,3,1) \longmapsto \begin{bmatrix} -\frac{4}{5}c_5 - \frac{1}{5} & -\frac{6}{5}c_5 + \frac{1}{5} & \frac{1}{2} \\ \frac{6}{5}c_5 + \frac{4}{5} & \frac{4}{5}c_5 + \frac{1}{5} & \frac{1}{2} \\ \frac{8}{5}c_5 + \frac{2}{5} & -\frac{8}{5}c_5 - \frac{2}{5} & 0 \end{bmatrix},$$

where $c_5 := \cos\frac{2\pi}{5}$. The fixed points and lines of the action are computed using the eigenspaces of the elements of $\rho(A_5)$. We obtain the following result:

- 6 points whose isotropy group is D_{10}.
- 10 points whose isotropy group is D_6.
- 12 points whose isotropy group is μ_5.

- 15 points whose isotropy group is μ_2^2.
- 20 points whose isotropy group is μ_3.
- 15 lines whose generic points (two of each type D_6, D_{10}, μ_2^2) have isotropy group μ_2.

By duality, the fixed lines are in orbits of line arrangements L_6, L_{10}, L_{15}, and L_{20} (the subindex indicates the number of lines) defined by $f_m(x,y,z)=0$. Note that $L_{10}+L_6$ is the icosididodecahedron arrangement; L_{15} is the closure of the points with isotropy group μ_2.

In the same way, we obtain a conic invariant by the action:

$$C_2 : f_2(x,y,z) := x^2 + (1-2c_5)y^2 + \frac{1-c_5}{2}z^2 = 0.$$

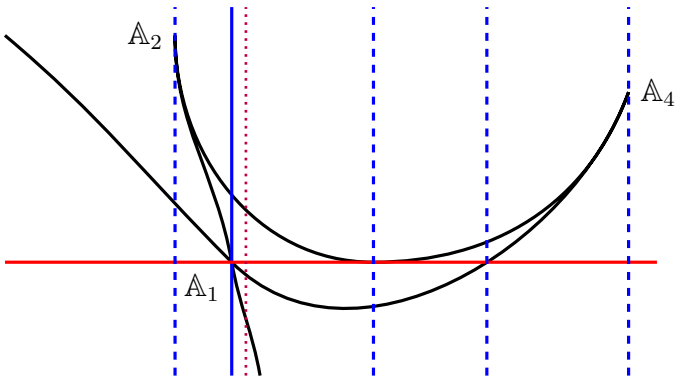
The map

$$\begin{array}{ccc} \mathbb{P}^2 & \xrightarrow{\pi} & \mathbb{P}^2_\omega \\ [x:y:z] & \longmapsto & [f_2 : f_6 : f_{10}]_\omega, \end{array}$$

where $\omega := (1,3,5)$ realizes the quotient. Under this map the image of the branching locus L_{15} is the curve C_{15}:

$$\begin{aligned} F_{15} :&= z^3 + (65y - 4x^3)x^2z^2 - 5(144y^2 - 159x^3y + 40x^6)xyz \\ &+ (1728y^3 - 3440x^3y^2 + 2275x^6y + 500x^9)y^2 = 0. \end{aligned}$$

This curve has a node at $[0:0:1]_\omega$ (smooth point). It has an $\mathbb{A}_2$-singularity at $\left[1 : -\frac{5}{27} : \frac{625}{81}\right]$, and it has an $\mathbb{A}_4$-singularity at $[1:1:1]_\omega$. Note that it is also tangent to $\{z=0\}$ along $\left[1:\frac{5}{8}:0\right]_\omega$. The real picture in the orbifold chart $z=1$.



For the computation of braid monodromy we put the generic line between the tacnode and the vertical line. The braid associated to the vertical line is $(\sigma_1 \cdot \sigma_2)^3$ and the relations are

$$[x_5 \cdot x_1 \cdot x_2 \cdot x_3, x_1] = [x_5 \cdot x_1 \cdot x_2 \cdot x_3, x_2] = [x_5 \cdot x_1 \cdot x_2 \cdot x_3, x_3] = [x_5, x_4] = 1.$$

The braid for the cusp is $(\sigma_1 \cdot \sigma_2 \cdot \sigma_1)^{-1} \cdot \sigma_3^3 \cdot (\sigma_1 \cdot \sigma_2 \cdot \sigma_1) = (\sigma_2 \cdot \sigma_1)^{-1} \cdot \sigma_3^3 \cdot (\sigma_2 \cdot \sigma_1)$. We obtain a cusp relation for $x_3^{\sigma_2 \cdot \sigma_1} = x_1$ and $x_4^{\sigma_2 \cdot \sigma_1} = x_4$:

$$x_1 \cdot x_4 \cdot x_1 = x_4 \cdot x_1 \cdot x_4.$$

For the tacnode we have σ_3^4 and the relation is

$$(x_3 \cdot x_4)^2 = (x_4 \cdot x_3)^2.$$

For the node we have $\sigma_3^{-2} \cdot \sigma_2^2 \cdot \sigma_3^2$ and the relation is

$$[x_2, x_4^{-1} \cdot x_3 \cdot x_4] = 1.$$

The last braid is $(\sigma_2 \cdot \sigma_3^2)^{-1} \cdot \sigma_3^5 \cdot (\sigma_2 \cdot \sigma_3^2) = \sigma_3^{-1} \cdot \sigma_2^5 \cdot \sigma_3$ giving

$$(x_3 \cdot x_4 \cdot x_3^{-1} \cdot x_2 \cdot (x_3 \cdot x_4 \cdot x_3^{-1}) \cdot x_2 \cdot (x_3 \cdot x_4 \cdot x_3^{-1}) = x_2 \cdot (x_3 \cdot x_4 \cdot x_3^{-1}) \cdot x_2 \cdot (x_3 \cdot x_4 \cdot x_3^{-1}) \cdot x_2.$$

We can replace this relation by

$$x_4 \cdot x_2 \cdot x_4 \cdot x_2 \cdot x_4 = x_2 \cdot x_4 \cdot x_2 \cdot x_4 \cdot x_2.$$

There are two more relations to be added. One is $x_1^2 = 1$ (which implies $x_2^2 = x_4^2 = 1$). For the other one, we have to look at infinity. We are projecting from the 5-singular point. Let us perform a weighted blow-up to obtain a singular ruled surface.

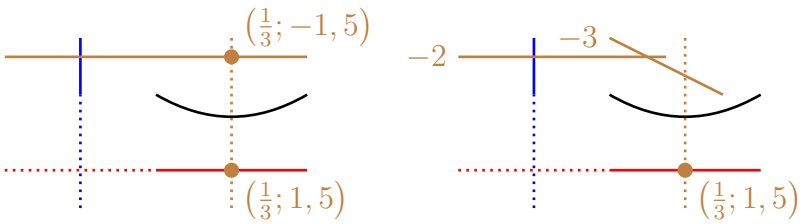


Figure 4. Singular ruled surface and resolution near the section at infinity

Let us recall that Zariski-van Kampen method is constructed in the classical case in a ruled surface. In this case, we construct it in a singular ruled surface. Note that the projection point is in our curve, so we have right to blow-up the weighted projective orbifold in order to produce the ruled surface. We denote by e a meridian of the section at infinity in the generic fiber where $x_1, \ldots, x_4$ live. Then, $x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot e = 1$. After the resolution we have introduced another curve. Actually the meridian ∞ of this curve is the one for which we may use the braid monodromy at infinity; it is not a trivial curve, but it will have some relations. Actually, from Mumford's ideas, $x_5 \cdot \infty = e^2$. We can express these elements in terms of the generators:

$$e = (x_1 \cdot x_2 \cdot x_3 \cdot x_4)^{-1}, \quad \infty = x_5^{-1} \cdot (x_1 \cdot x_2 \cdot x_3 \cdot x_4)^{-2}.$$

Note that e and x_5 commute. They also satisfy $e = \infty^3$. Hence,

$$(x_1 \cdot x_2 \cdot x_3 \cdot x_4)^5 \cdot x_5^3 = 1.$$

Proposition 8.1. *Let $\mathcal{A}$ be the icosidodecahedron arrangement. and let $\pi : \mathbb{P}^2 \to \mathbb{P}^2_{A_5}$ be the orbifold map defined by the action of the automorphism group of $\mathcal{A}$. Then,*

$$\begin{aligned} \pi_1^{\mathrm{orb}}(\mathbb{P}^2_{A_5}) = \langle x_1, \ldots, x_5 | \, & x_1^2 = x_2^2 = x_4^2 = [x_5, x_4] = (x_1 \cdot x_4)^3 = 1, (x_3 \cdot x_4)^2 = (x_4 \cdot x_3)^2, \\ & (x_2 \cdot x_4)^5 = (x_1 \cdot x_2 \cdot x_3 \cdot x_4)^5 \cdot x_5^3 = [x_2, x_4 \cdot x_3 \cdot x_4] = 1, \\ & [x_2 \cdot x_3 \cdot x_5, x_1] = [x_3 \cdot x_5 \cdot x_1, x_2] = [x_5 \cdot x_1 \cdot x_2, x_3] = 1 \rangle \, . \end{aligned} \tag{8.1}$$

Let

$$\begin{aligned} \pi_1^{\mathrm{orb}}(\mathbb{P}^2_{A_5}) &\xrightarrow{\rho} A_5 \\ x_1 &\longmapsto (2,5)(3,4) \\ x_2 &\longmapsto (2,3)(4,5) \\ x_4 &\longmapsto (1,2)(3,4) \\ x_3, x_5 &\longmapsto () \end{aligned}$$

Then $\ker \rho = \pi_1(\mathbb{P}^2 \setminus \mathcal{A})$.

Remark 8.2. Let G be the quotient of $\pi_1^{\mathrm{orb}}(\mathbb{P}^2_{A_5})$ where we *kill* x_3^2, x_5^2 (the square of the meridians of the lines of $\mathcal{A}$) and $(x_3 \cdot x_4)^2, (x_1 \cdot x_2 \cdot x_4 \cdot x_5)^2$ (the meridians of the exceptional components of the blowing-up of the multiple points of $\mathcal{A}$). This group has order 240 and it is isomorphic to $\mathbb{A}_5 \times \mu_2^2$. The fundamental group of

a minimal smooth Z model of a double covering of $\mathbb{P}^2$ ramified at $\mathcal{A}$ is the kernel of this group by a map onto $A_5 \times \mu_2$ which is the group of two elements. This is behind the result of Yoshinaga concerning the torsion on the homology of the Milnor fiber of $\mathcal{A}$.

It is not hard to see that the $\chi(Z) = 64$ and $K_Z^2 = 20$.

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