

ORBIFOLDS

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INTRODUCTION

These series of lectures mainly focus in the study of some properties of orbifold fundamental groups and orbifold fundamental covers, ending with Seifert fibrations and more general orbifold fibre bundles, specially those related with elliptic fibrations. There are basically two ways to introduce the orbifold fundamental group: the loop-based and the cover-base. We follow the first one without forgetting the use of orbifold covers. We will start this discussion relating orbifold and ramified covers. There are many references in the literature, we advise the readers to take a look to some classical, [1, 4, 3], and also some recent ones [2, 6].

Let us start by pointing out several variants of orbifolds, which can be considered. We focus less on the geometric aspects. In dimension 2 they reappear naturally, and in dimension 3, they will be studied in further lectures.

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- (O1) Implicitly, orbifolds X have been defined without boundary, but the definition extends flawlessly to the boundary case (even to orbifolds with corners). Note that in this case the part of $\partial|X|$ corresponding to mirror symmetries is not part of ∂X .

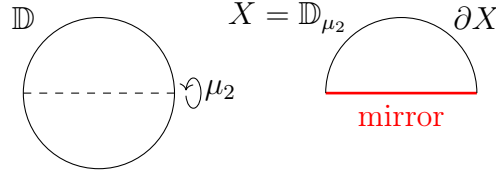


Figure 1. Orbifold with boundary and mirror, see Notation 1(3).

- (O2) Orbifold maps have been defined as differentiable, but it is possible to define continuous orbifold maps in the same way.
- (O3) Holomorphic orbifolds can be defined accordingly: the groups of the actions will be subgroups of analytic diffeomorphisms.
- (O4) A more delicate generalization can be done. We may replace the open sets of the charts by singular spaces with group actions. To keep some differentiable structure we may assume that the singular models are *nice* constructible subsets of some $\mathbb{R}^n$ and that the group actions are restriction of diffeomorphisms.

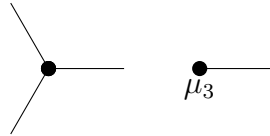


Figure 2. Singular orbifold

For example we may consider X as the orbifold defined as by the action of μ_3 (the group of cubic roots of unity in $\mathbb{C}$) by rotation on the space Z of Figure 2. The quotient would be a interval but one of the boundary points has isotropy group μ_3 .

Notation 1. Let us fix some notations:

- (1) The finite cyclic groups are represented either as $\mathbb{Z}/m$ or as μ_m , the group of m -roots of unity in $\mathbb{C}$.

- (2) The disk (resp. closed disk) in $\mathbb{C}$ of center z and radius r is represented as $\mathring{\mathbb{D}}_r(z)$ (resp. $\mathbb{D}_r(z)$); we may drop the notation for the center if $z = 0$ and for the radius if $r = 1$. We use the same criteria for circles: $\mathbb{S}_r^1(z)$. And also for n -dimensional balls or spheres in $\mathbb{R}^n$: $\mathring{\mathbb{B}}_r^n(\mathbf{x})$, $\mathbb{B}_r^n(\mathbf{x})$, $\mathbb{S}_r^{n-1}(\mathbf{x})$.
- (3) If X is a manifold with a faithful and properly discontinuous action of a group G , the induced orbifold structure in X/G will be denoted by X_G . If $X = \mathbb{C}$ and $G = \mu_m$ acting by multiplication of m -roots of unity, we will denote the orbifold as $\mathbb{C}_{(m)}$; in the same way we will denote $\mathbb{D}_{(m)}$ or $\mathring{\mathbb{D}}_{(m)}$. More generally an orbifold X such that $|X|$ is a smooth surface and the singular points are only of cyclic types $n_1, \dots, n_r$ will be denoted as $X = S_{(n_1, \dots, n_r)}$. The position of the points will be made explicit only if needed.
- (4) Unless otherwise stated we will consider right faithful discrete actions.
- (5) For an orbifold map $f : X \rightarrow Y$ we will denote by $|f| : |X| \rightarrow |Y|$ the underlying map.
- (6) If M is an oriented manifold with boundary, the induced orientation on the boundary ∂M , is the one such that adding an exterior normal vector, we obtain the orientation of M .

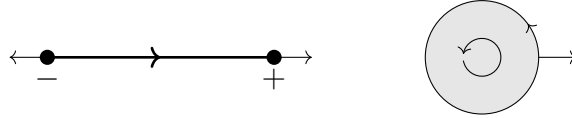


Figure 3. Orientation on the boundary

1. RAMIFIED COVERS AND ORBIFOLD COVERS

We are going to give an interpretation of ramified covers in terms of orbifolds. We start with ramified covers in dimension 2.

Example 1.1. Let us consider a map $p : \mathbb{C} \cup \{\infty\} \equiv \mathbb{P}^1 \rightarrow \mathbb{P}^1$ defined by the polynomial:

$$p(t) := 6t^5 - 45t^4 + 130t^3 - 180t^2 + 120t + 1, \quad p(\infty) = \infty.$$

This is a ramified 5-cover; since $p'(t) = 30(t-1)^2(t-2)^2$, the branching points are $t = 1, 2, \infty$ and the branching values are $p(1) = 32$, $p(2) = 33$, and ∞ . It is easily seen that

$$p^{-1}(\infty) = \{\infty\}, \quad p(t)-32 = (t-1)^3(6t^2-27t+31), \quad p(t)-33 = (t-2)^3(6t^2-9t+4).$$

Example 1.2. Let $m, n, p \in \mathbb{N}$ such that $m = np$. The following diagram shows how to construct an orbifold cover $\mathbb{C}_{(n)} \rightarrow \mathbb{C}_{(m)}$:

$$\begin{array}{ccccc}
 z & \xrightarrow{\quad} & z & & \\
 \downarrow & \mathbb{C} \xrightarrow{1_{\mathbb{C}}} \mathbb{C} & \downarrow & & \\
 [z] & \mathbb{C}/\mu_n \xrightarrow{|\varphi_{n,m}|} \mathbb{C}/\mu_m & [z] & & \\
 \downarrow & \downarrow \cong & \downarrow \cong & & \\
 & \mathbb{C} \xrightarrow{\quad} \mathbb{C} & & & \\
 \downarrow & w \xrightarrow{\quad} w^p & \downarrow & & \\
 z^n & \xrightarrow{\quad} & z^m & &
 \end{array}$$

In case $n = 1$, we can see the orbifold cover as a model of the usual $m : 1$ branched cover of $\mathbb{C}$, ramified at 0.

Let S be a surface; for simplicity we assume that S is obtained from a compact closed surface $\bar{S}$ taking out a finite number of pairwise open disks and a finite number of isolated points.

Definition 1.3. A proper continuous map $\pi : \tilde{S} \rightarrow S$, $S, \tilde{S}$ surfaces as above, is a *finite ramified cover* if there is a finite set $F \subset S$ of interior points such that $\pi| : \tilde{S} \setminus \pi^{-1}(F) \rightarrow S \setminus F$ is a cover.

Let us recall that, with the above notation, for each $p \in F$ there is a closed neighbourhood $\mathbb{D}_p$ of p (homeomorphic to a disk) such $\pi^{-1}(p) = \bigsqcup_{q \in \pi^{-1}(p)} \tilde{\mathbb{D}}_q$, where $\tilde{\mathbb{D}}_q$ is a closed neighbourhood of q (homeomorphic to a disk) and the following diagram holds

$$\begin{array}{ccc}
 \tilde{\mathbb{D}}_q & \xrightarrow{\pi|} & \mathbb{D}_p \\
 \downarrow \Phi_q & & \downarrow \Phi_p \\
 \mathbb{D} & \xrightarrow{\quad} & \mathbb{D} \\
 z & \xrightarrow{\quad} & z^{n_q};
 \end{array}$$

under these conditions, π ramifies at q with ramification index $n_q - 1$.

Under these conditions we are going to consider orbifold structures S_π (resp. $\tilde{S}_\pi$) on S (resp. $\tilde{S}$) and construct an orbifold map $\Pi : \tilde{S}_\pi \rightarrow S_\pi$ such that $|\Pi| = \pi$.

The set of singular points of S_π is F : each $p \in F$ is a conic point of order $m_p := \text{lcm}\{n_q \mid q \in \pi^{-1}(p)\}$, i.e. of type $\mathbb{D}_{(m_p)}$.

The set of singular points of $\tilde{S}_\pi$ is contained in $\pi^{-1}(F)$. Let $q \in \pi^{-1}(F)$; it is a conic point of order $\tilde{m}_q := \frac{m_{\pi(q)}}{n_q}$, i.e., of type $\mathbb{D}_{(\tilde{m}_q)}$. In particular, if π is a Galois cover, $\tilde{S}_\pi = \tilde{S}$.

Let us check that the cover property is satisfied:

$$\begin{array}{ccc}
 z & \xrightarrow{\quad} & z \\
 \downarrow & \begin{array}{ccc} \mathbb{D} & \xrightarrow{\mathbf{1}_{\mathbb{D}}} & \mathbb{D} \\ \downarrow & & \downarrow \\ \tilde{\mathbb{D}}_q & \xrightarrow{|\Pi|=\pi|} & \mathbb{D}_p \\ \downarrow \Phi_q & & \downarrow \Phi_p \\ \mathbb{D} & \xrightarrow{\quad} & \mathbb{D} \\ w & \xrightarrow{\quad} & w^{n_q} \end{array} & \downarrow \\
 z^{\frac{m_p}{n_q}} & \xrightarrow{\quad} & z^{m_p}.
 \end{array}$$

Example 1.4. Let us continue with Example 1.1. The unique preimage for ∞ is ∞ , hence $m_\infty = n_\infty = 5$. For 32 (resp. 33) we have 3 preimages; two of them are unramified and the third one satisfies $n_1 = 3$ (resp. $n_2 = 3$) and $m_{32} = m_{33} = 1$. Then, we can interpret p as an orbifold cover $\mathbb{S}_{(3,3,3,3)}^2 \rightarrow \mathbb{S}_{(3,3,5)}^2$.

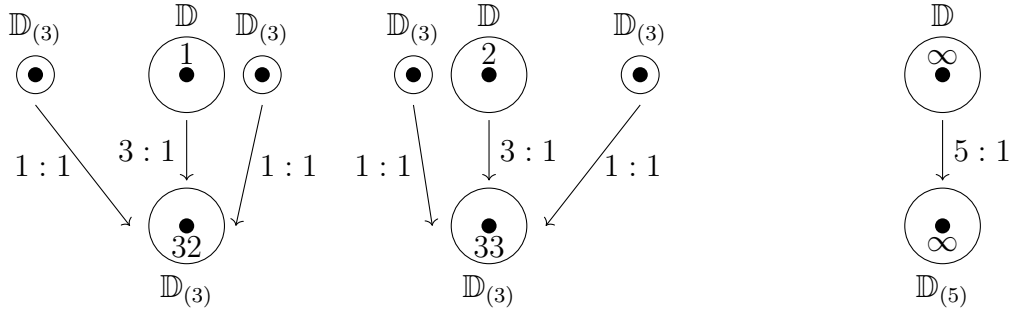


Figure 4. Singular points and ramifications

We are exploiting the fact that topologically conic points are smooth points. This approach may work for ramified covers in higher dimensions. Let us consider now a Galois orbifold cover $\pi : \mathbb{S}^2 \rightarrow S$ where S is an orbifold. These maps are classified and we have:

$$(1.1) \quad S = \mathbb{S}_{(n,n)}^2, \mathbb{S}_{(p,q,r)}^2, \quad (p,q,r) = (2,2,n), (2,3,3), (2,3,4), (2,3,5).$$

$n > 1$ $n > 1$

In all these cases taking cones we can construct orbifolds $\overline{\mathbb{B}}_G^3$ such that $\overline{\mathbb{B}}^3/G$ is homeomorphic to a closed ball, i.e., it is topologically regular. Let us describe them:

- (G1) $G = \mu_n$. The singular points are a diameter of the ball and the isotropy groups for these points is μ_n . This is the local situation of ramified covers of a 3-manifold along a link.
- (G2) $G = \langle a, b, c \mid a^p = b^q = c^r = abc = 1 \rangle$. The singular points are the cone of three points in the sphere; at each edge the isotropy groups are μ_p, μ_q, μ_r and at the center the isotropy group is G .

Hence ramified covers of three manifolds where the ramification locus is a trivalent graph with suitable ramification indices at the trivalent vertices can be interpreted as orbifold covers.

Example 1.5. In dimension 4, the ramified cover $\mathbb{C}^2 \rightarrow \mathbb{C}^2, (x, y) \mapsto (x^n, y^m)$ can also be interpreted as an orbifold cover $\mathbb{C}^2 \rightarrow \mathbb{C}_{(n)} \times \mathbb{C}_{(m)}$.

Example 1.6. We can give explicit examples for the ramified covers $\mathbb{S}^2 \rightarrow \mathbb{S}^2$ of (1.1), identifying $\mathbb{S}^2 \equiv \mathbb{P}^1 \equiv \mathbb{C} \cup \{\infty\}$. For the case (n, n) it is easy, taking $f_{(n,n)}(t) = t^n$; $0, \infty$ are both the critical points and values. For the dihedral case $(2, 2, n)$, it is well-known that we can take $f_{(2,2,n)} = t^n + t^{-n}$. Note that this cover factorizes taking first a double cover and then an n -cyclic cover. Let us assume

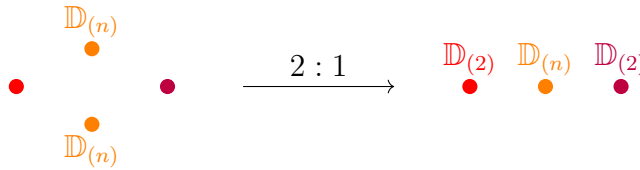


Figure 5. Factorization of the dihedral cover.

that the two points of order 2 are ± 2 and that 0 is the point of order n . Then,

$t \mapsto t + t^{-1}$ is a double cover with discriminant points ± 2 , and the preimage of 0 is $\{0, \infty\}$. Similar ideas are used to find the cover for $(2, 3, 3)$. We factorize through a triple cover ramified at the two points of order 3; the preimage of the point of order 2 gives three points of order 2 and we apply a $(2, 2, 2)$ cover.

$$f_{(2,3,3)} = \left(\frac{3t^4 + 6t^2 - 1}{3t^4 - 6t^2 - 1} \right)^3.$$

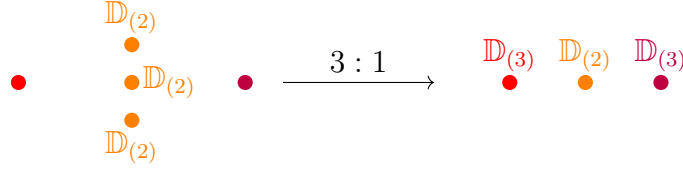


Figure 6. Factorization for $(2, 3, 3)$.

A similar idea is used for $(2, 3, 4)$, obtaining the polynomial

$$f_{(2,3,4)} = \left(\frac{t^2 \cdot (3t^4 + 1)^2}{(3t^4 - 1) \cdot (9t^8 + 102t^4 + 1)} \right)^2.$$

We start with a double cover ramified at the points of order 2, 4. The preimage of the point of order 4 has order 2 and we get two points of order 3 from the preimages of the point of order 3.

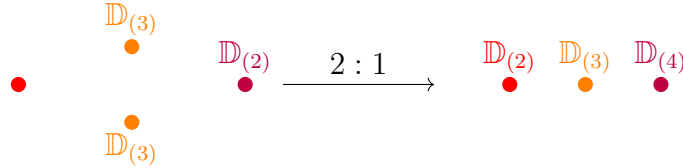


Figure 7. Factorization for $(2, 3, 4)$.

The case $(2, 3, 5)$ is a more involved. We start with a non-Galois cover of degree 5, e.g., the one corresponding to $h(t) := t^5 + 5t^4 + 40t^3$. This map ramifies with order 5 at ∞ and $h'(t) = 5t^2(t^2 + 4t + 24)$. The two critical values have the

same image. One example of such polynomials is:

$$\begin{aligned}
 f_{(2,3,5)} &= \frac{(p_1 \cdot p_2 \cdot p_3)^3}{(q_1 \cdot q_2)^5} \\
 p_1 &= 3s^4 + 10s^2 + 45 \\
 p_2 &= 2s^8 - 5s^6 + 290s^4 - 75s^2 + 450 \\
 p_3 &= s^8 + 80s^6 - 790s^4 + 1200s^2 + 225 \\
 q_1 &= s^4 - 4s^2 + 15 \\
 q_2 &= s^8 - 30s^6 - 350s^4 - 450s^2 + 225.
 \end{aligned}$$

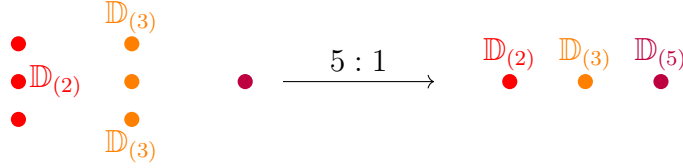


Figure 8. Factorization for $(2, 3, 5)$.

2. ORBIFOLD FUNDAMENTAL GROUP

The goal is to interpret the orbifold fundamental group of an orbifold as a group of (orbifold) homotopy classes of (orbifold) loops as in the classical case. Let us see some particular properties of orbifold maps.

Example 2.1. Let X, Y be orbifolds and let $f : |X| \rightarrow |Y|$ be a continuous map. How many orbifold maps $F : X \rightarrow Y$ satisfy that $f = |F|$?

- (1) Let $X := \mathbb{R}$ with *smooth* orbifold structure ($X = |X|$), and let $Y := \mathbb{R}_{\mu_2}$ the orbifold obtained from the antipodal action of μ_2 on $\mathbb{R}$, where $|Y| = \mathbb{R}/\mu_2$ is identified with $[0, \infty)$. Let us consider the continuous map $f : X \rightarrow |Y|$ defined by $f(t) := |t|$. We can define two orbifold maps $F_1, F_2 : X \rightarrow Y$ such that $|F_1| = f = |F_2|$. Let $\tilde{F}_i : \mathbb{R} \rightarrow \mathbb{R}$ be the corresponding lifting. Then, $\tilde{F}_1(t) := |t|$, and $\tilde{F}_2(t) := t$. The map F_1 is orbifold homotopy equivalent to a constant map, while F_2 is not.
- (2) Let $X := \mathbb{C}$ with *smooth* orbifold structure, and let $Y := \mathbb{C}_{(m)}$. Let us consider the map $f : |Y| \rightarrow |X|$, given by $f([z]) = z^m$. This is actually a homeomorphism and there is a unique orbifold map $F : Y \rightarrow X$ such that

$|F| = f$; the lifting is

$$\begin{array}{ccc}
 \mathbb{C} & & \\
 \downarrow \pi & \nearrow \tilde{f} & \\
 [z] & \xrightarrow{\quad} & z^m \\
 \downarrow & \searrow f & \\
 |Y| & \xrightarrow{\quad} & X
 \end{array}$$

Since f is a homeomorphism, $g := f^{-1}$ is well-defined but there is no orbifold map $G : X \rightarrow Y$ such that $|G| = g$ since it clearly does not admit a lift to $\mathbb{C}$:

$$\begin{aligned}
 \mathbb{C} &\equiv X \xrightarrow{f^{-1}} |Y| \equiv \mathbb{C}/\mu_m \\
 z &\longmapsto [\sqrt[m]{z}].
 \end{aligned}$$

- (3) Let X be an orbifold such that $|X|$ is a manifold and all the isotropy groups preserve orientations. Then, there is a unique orbifold map $f : X \rightarrow |X|$ such that $|f| = \mathbf{1}_{|X|}$ and it is called the *forget map*. This example generalizes (2). In that cases the identity define also a partial forget orbifold map $\mathbb{C}_{(n)} \rightarrow \mathbb{C}_{(m)}$ when n is a divisor of m . It is bijective but it is not an orbifold isomorphism (if $m \nmid n$) since the inverse cannot define an orbifold map.

As in the standard case we may have two ways to define the orbifold fundamental group. Either we consider orbifold homotopy classes of orbifold loops or we consider isomorphisms on the *universal* orbifold cover. Which properties should orbifold fundamental groups satisfy?

- (P1) If X is a smooth manifold, it should coincide with the usual fundamental group.
- (P2) Let X be a manifold with a faithful and properly discontinuous action of a discrete group G . The group will fit in an exact sequence

$$1 \longrightarrow \pi_1(X) \longrightarrow \pi_1^{\text{orb}}(X_G) \longrightarrow G \longrightarrow 1.$$

In particular, if X is simply connected, $\pi_1^{\text{orb}}(X_G) \cong G$.

- (P3) The above property should work also for discrete actions on orbifolds.
- (P4) The relationship with orbifold covers should be similar to the classical case.

Definition 2.2. Let X be an orbifold.

- (i) An *orbifold path* is an orbifold continuous map $\gamma : [0, 1] \rightarrow X$.
- (ii) An *orbifold loop* is an orbifold map whose end points coincide.
- (iii) An *orbifold homotopy relative to the end points* between two orbifold paths $\gamma_0, \gamma_1 : [0, 1] \rightarrow X$ is an orbifold continuous map $H : [0, 1] \times [0, 1] \rightarrow X$ such that the restriction of H to the boundary is as in Figure 9.

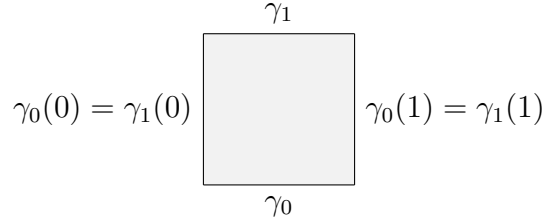


Figure 9

Example 2.3. Let $X = \mathbb{D}_{(2)}$ and let us consider the unique orbifold loop γ_1 supported by $|\gamma_1| : [0, 1] \rightarrow \mathbb{D}/\mu_2$, given by $|\gamma_1|(t) := [\exp i\pi t]$, and let γ_0 be the constant map to 1. It is obvious that $|\gamma_0|$ and $|\gamma_1|$ have the same homotopy type in $\mathbb{D}/\mu_2$.

Since a lifting of the orbifold map γ_0 to $\mathbb{C}$ is the constant map to 1, and a lifting of the orbifold map γ_1 to $\mathbb{C}$ is not a closed map, it is not possible to construct an orbifold homotopy between γ_0 and γ_1 .

Definition 2.4 (Loop-based orbifold fundamental group). Let X be an orbifold, and let $x \in |X|$ be a regular point. The *orbifold fundamental group* $\pi_1^{\text{orb}}(X; x)$ of X at x is the group of orbifold homotopy classes (relative to the end points) of orbifold loops based at x .

It is obvious that (P1) is satisfied.

Proposition 2.5. *This definition satisfies (P2).*

Proof. Let X be a simply connected manifold and let G be a group acting discretely and faithfully. We are going to check that G is isomorphic to $\pi_1^{\text{orb}}(X_G; x)$ for some $x \in X/G$ with trivial isotropy group. Let us fix a representative $\tilde{x} \in X$ of x .

Let $\gamma : [0, 1] \rightarrow X$ be an orbifold loop based on x , which is determined in this case by liftings $\gamma_{\tilde{x}} : [0, 1] \rightarrow X$ such that $\tilde{\gamma}(0) = \hat{x}$, for any representative $\tilde{x}$ of x . Let $g \in G$ be the unique element such that $\tilde{x}^g = \gamma_{\tilde{x}}^-(1)$. Since X is simply

connected, it is not hard to see that g depends only on the class of γ in $\pi_1^{\text{orb}}(X_G; x)$. We need to check that the induced map $\pi_1^{\text{orb}}(X_G; x) \rightarrow G$ is a group isomorphism. It is sketched in Figure 10, following the same ideas as in free actions. Note that

$$(\gamma \cdot \delta)_{\tilde{x}}^- = (\delta^- \cdot \gamma^-)_{\tilde{x}} = \delta_{\tilde{x}}^- \cdot \gamma_{\tilde{x}^h}^- = \delta_{\tilde{x}}^- \cdot (\gamma_{\tilde{x}}^-)^h.$$

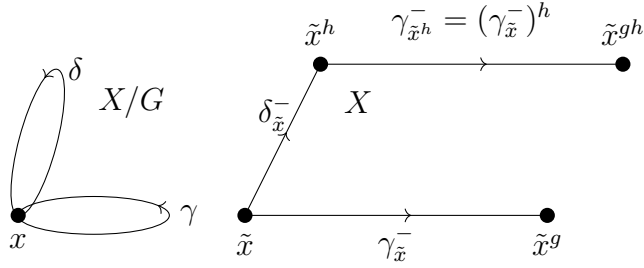


Figure 10. Liftings and compositions

As expected, the isomorphism depends on the choice of $\tilde{x}$ and it is only well-defined up to conjugation. $\square$

Examples 2.6. Let us see several important examples which will be used later.

- (a) $\pi_1^{\text{orb}}(\mathbb{C}_{(m)}; [\exp \frac{i\pi}{m}]) \cong \mathbb{Z}/m \cong \mu_m$ and a generator is given by the class of $\gamma : [0, 1] \rightarrow \mathbb{C}/\mu_m$ given by $\gamma(t) := [\exp(\frac{2t-1}{m}i\pi)]$ with the only suitable lifting. The same holds for $\mathbb{D}_{(m)}$.

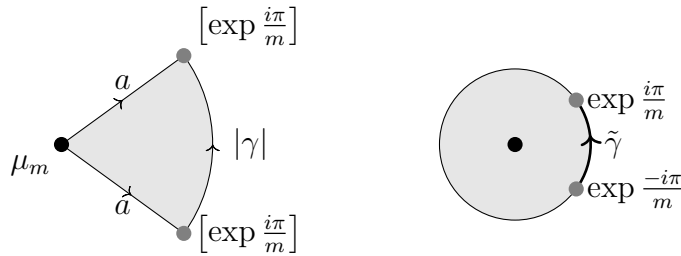


Figure 11. $\pi_1^{\text{orb}}(\mathbb{C}_{(m)}; [\exp \frac{i\pi}{m}]) \cong \mu_m$.

- (b) $\pi_1^{\text{orb}}(\mathbb{R}_{\mu_2}; [1])$ is generated by the *Alice map* $\gamma : [0, 1] \rightarrow \mathbb{R}_{\mu_2}$ defined by the lifting that *crosses the mirror*:

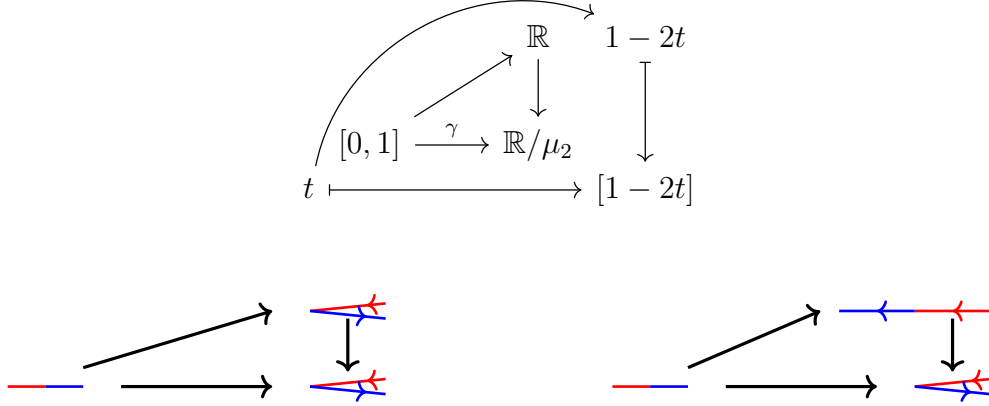


Figure 12. The two orbifold maps supported by $|\gamma|$.

- (c) Let us consider the dihedral action of D_{2m} on $\mathbb{C}$, defined by rotation and conjugation. Then, $\pi_1^{\text{orb}}(\mathbb{C}_{D_{2m}}; [1]) \cong D_{2m}$. The classes of the Alice maps $\gamma_{\pm} : [0, 1] \rightarrow \mathbb{C}_{D_{2m}}$ such that $|\gamma_{\pm}|(t) := [\exp \frac{\pm i\pi}{m}]$ are two order-2 elements generating the fundamental group. Note that the cover $\mathbb{C} \rightarrow \mathbb{C}_{D_{2m}}$ factorizes through $\mathbb{C} \rightarrow \mathbb{C}_{(m)} \rightarrow \mathbb{C}_{D_{2m}}$.

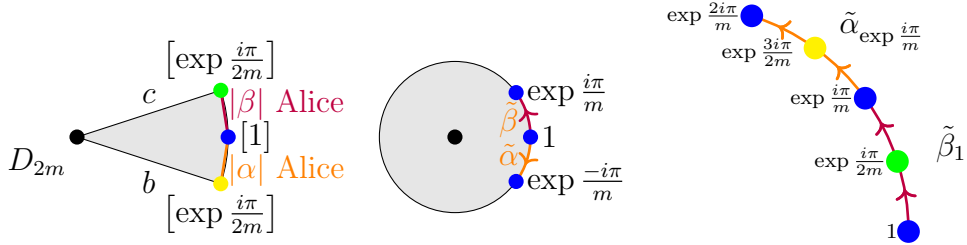


Figure 13. The liftings of α^2 and β^2 are closed. The lifting of $\beta \cdot \alpha$ is $\tilde{\gamma}$ in Figure 11, and then the lifting of γ^m is also closed.

Most properties of fundamental groups have a direct translation to orbifold fundamental groups. For example, it is functorial for pointed orbifold maps (continuity is enough). For its computation we still have Seifert-van Kampen Theorem which the same statement and proof.

Theorem 2.7. *Let X be an orbifold obtained by the union of two open orbifolds U, V such that $U \cap V$ decomposes in orbifold connected components $\{W_i\}_{i \in I}$. Fix one index $0 \in I$.*

Let us choose $x_i \in W_i$ for each $i \in I$ and paths $\gamma_{U,i} : [0, 1] \rightarrow U$, $\gamma_{V,i} : [0, 1] \rightarrow V$ from x_0 to x_i for each $i \in I \setminus \{0\}$. Let us denote $\delta_i := [\gamma_{V,i} \cdot \gamma_{U,i}^{-1}]$. Then,

$$\pi_1^{\text{orb}}(X; x_0) \cong \frac{\pi_1^{\text{orb}}(U; x_0) * \pi_1^{\text{orb}}(V; x_0) * \langle \gamma_i, i \in I \setminus \{0\} \mid _ \rangle}{R}$$

where R is the normal subgroup generated by

$$\{(j_U)_*(g) \cdot (j_V)_*(g)^{-1} \mid g \in \pi_1^{\text{orb}}(W_0; x_0)\}$$

and

$$\{\delta_i \cdot (j_{i,U})_*(g) \cdot \delta_i^{-1} \cdot (j_{i,V})_*(g)^{-1} \mid g \in \pi_1^{\text{orb}}(W_i; x_i)\}, \quad i \in I \setminus \{0\},$$

where

$$\begin{array}{ccc} \pi_1^{\text{orb}}(W_i; x_i) & \xrightarrow{(j_{i,U})_*} & \pi_1^{\text{orb}}(U; x_0) & \pi_1^{\text{orb}}(W_i; x_i) & \xrightarrow{(j_{i,V})_*} & \pi_1^{\text{orb}}(V; x_0) \\ g & \longmapsto & \gamma_{U,i} \cdot (j_U)_*(g) \cdot \gamma_{U,i}^{-1} & g & \longmapsto & \gamma_{V,i} \cdot (j_V)_*(g) \cdot \gamma_{V,i}^{-1} \end{array}$$

We recall the two usual versions of this theorem. If $U \cap V$ is connected, we have an amalgamated product; if $U \cap V$ has two connected components which are simply connected then the fundamental group is the free product of the two factors and $\mathbb{Z}$.

Example 2.8. Let us use these results to compute the orbifold fundamental group of a generalization of the billiard orbifold. Let P_n be an orbifold supported by polygon of n sides. The singular locus of P_n is in the boundary. Each open side has isotropy group $\mathbb{Z}/2$, while the vertices have isotropy group the Klein group $\mathbb{Z}/2 \times \mathbb{Z}/2$. We can decompose P_n in two open sets U, V , where U is the interior of the polygon and V is an open neighbourhood of the boundary. The orbifold V is decomposed in n orbifolds V_i supported by open neighbourhoods of the vertices. It is easily seen that $\pi_1^{\text{orb}}(V_i) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, generated by Alice paths to the adjacent edges. Let us denote these paths by γ_i . By induction we see that

$$\pi_1^{\text{orb}}(V_1 \cup \dots \cup V_{n-1}) = \left\langle \gamma_i \mid \gamma_i^2 = 1, [\gamma_i, \gamma_{i+1}] = 1 \right\rangle.$$

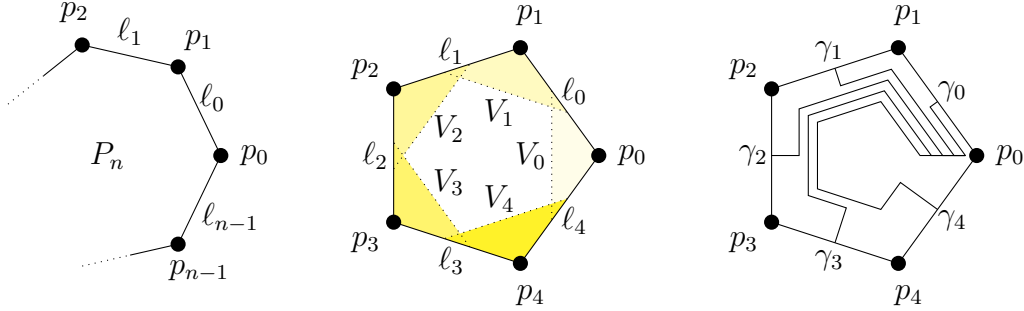


Figure 14. Right-angled orbifold polygon. The isotropy groups at p_i are $\mathbb{Z}/2 \times \mathbb{Z}/2$ and at ℓ_i are $\mathbb{Z}/2$.

The intersection of $V_1 \cup \dots \cup V_{n-1}$ and V_n has two connected components. If δ is path *parallel* to the boundary we obtain that

$$\pi_1^{\text{orb}}(V) = \left\langle \delta, \gamma_i \left| \begin{array}{l} \gamma_i^2 = 1, [\gamma_i, \gamma_{i+1}] = 1, [\delta^{-1} \cdot \gamma_{n-1} \cdot \delta, \gamma_0] = 1 \\ i \in \mathbb{Z}/n \end{array} \right. \right\rangle.$$

Since U is simply connected, we have

$$(2.1) \quad \pi_1^{\text{orb}}(P_n) = \left\langle \gamma_i \left| \begin{array}{l} \gamma_i^2 = [\gamma_i, \gamma_{i+1}] = 1 \\ i \in \mathbb{Z}/n \end{array} \right. \right\rangle.$$

The double orbifold cover of P_n given by the mirrors is homeomorphic to $H_n := \mathbb{S}_{(2_1, \dots, 2_n)}^2$. Its fundamental group is

$$\pi_1^{\text{orb}}(H_n) = \left\langle \delta_i \left| \begin{array}{l} \delta_i^2 = 1, \prod_{i=0}^{n-1} \delta_i = 1 \\ i \in \mathbb{Z}/n \end{array} \right. \right\rangle.$$

They satisfy the following exact sequence:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_1^{\text{orb}}(H_n) & \longrightarrow & \pi_1^{\text{orb}}(P_n) & \longrightarrow & \mu_2 \longrightarrow 1 \\ & & & & \gamma_i & \longmapsto & -1 \\ & & \delta_i & \longmapsto & \gamma_i \cdot \gamma_{i+1} & & \end{array}$$

We see two possible generalizations of this example. In the first one we replace right angles by angles $\frac{2\pi}{m_i}$ and in the second one, we consider higher dimensions.

Example 2.9. Since $\mathbb{Z}/2 \times \mathbb{Z}/2$ can be seen as the dihedral group of a bigon, we can consider another polygon $P_n(m_0, \dots, m_{n-1})$ where the isotropy group at ℓ_i is $\mathbb{Z}/2$ and the isotropy group at p_i is D_{2m_i} . Then

$$\begin{aligned} \pi_1^{\text{orb}}(P_n(m_0, \dots, m_{n-1})) &= \left\langle \gamma_i \left| \gamma_i^2 = \mathbb{A}_{m_i}(\gamma_{i-1}, \gamma_i) = 1 \right. \right\rangle \\ &= \left\langle \gamma_i \left| \gamma_i^2 = (\gamma_{i-1} \cdot \gamma_i)^{m_i} = 1 \right. \right\rangle \end{aligned}$$

where

$$\mathbb{A}_m = (x \cdot y \cdot \dots) \cdot (y \cdot x \cdot \dots)^{-1},$$

$m \text{ terms} \qquad m \text{ terms}$

the relation of a torus link of type $(m_i, 2)$. As in Example 2.8 there is a double orbifold cover of $P_n(m_0, \dots, m_{n-1})$ whose domain is $\mathbb{S}_{(m_0, \dots, m_{n-1})}^2$.

Example 2.10. Let P be a polytope such that all its vertices are simple, i.e., they are the intersection of n facets (faces of codimension 1). We can consider an orbifold X_P such that $|X_P| = P$, and such that the points in the faces of codimension k have $(\mathbb{Z}/2)^k$ as isotropy group.

Using Seifert-van Kampen theorem, we obtain a presentation of the fundamental group such that the orbifold fundamental group is generated by γ_F , for F facet of P , and two types of relations. For any F , we have $\gamma_F^2 = 1$, and for non-disjoint facets F, G we have $[\gamma_F, \gamma_G] = (\gamma_F \cdot \gamma_G)^2 = 1$.

3. ORBIFOLD COVERS

Definition 3.1. Let X be an orbifold. We say that $p : \tilde{X} \rightarrow X$ is a *universal orbifold cover* if it is an orbifold cover and for each orbifold cover $q : \hat{X} \rightarrow X$, there is an orbifold cover $h : \tilde{X} \rightarrow \hat{X}$ such that $p = q \circ h$, i.e., the following diagram commutes:

$$\begin{array}{ccc} & \tilde{X} & \\ & \swarrow h & \downarrow p \\ \hat{X} & \xrightarrow{q} & X \end{array}$$

Definition 3.2. Let $q : \hat{X} \rightarrow X$ be an orbifold cover. The *deck group* of q is the group of orbifold diffeomorphisms $h : \hat{X} \rightarrow \hat{X}$ such that $q = q \circ h$. Its elements are called the *deck transformations* of q .

The definition of orbifold map has some extra details in the case of diffeomorphisms.

Lemma 3.3. *Let $h : X \rightarrow Y$ be an orbifold diffeomorphism, $x \in |X|$. The groups isotropy groups Γ_x and $\Gamma_{|h|(x)}$ are isomorphic via the equivariance of the lifting and such an isomorphism is well defined up to conjugation.*

Proof. Let $x \in |X|$ and let $y := |h|(x) \in |Y|$. Fix orbifold charts $(\tilde{U}_x, U_x, \Gamma_x, \Psi_x)$, $x \in U_x$, and $(\tilde{U}_y, U_y, \Gamma_y, \Psi_y)$, $y \in U_y$, such that $|h|(U_x) = U_y$, and Γ_x, Γ_y are the isotropy groups of x, y , respectively. Fix a lifting $\tilde{h} : \tilde{U}_x \rightarrow \tilde{U}_y$ relative to these charts. The diffeomorphism property implies that given $\gamma_x \in \Gamma_x$, there exists a unique $\gamma_y \in \Gamma_y$ such that $\tilde{h} \circ \gamma_x = \gamma_y \circ \tilde{h}$. The map

$$\begin{aligned} \Gamma_x &\xrightarrow{\Theta_{\tilde{h}}} \Gamma_y \\ \gamma_x &\longmapsto \tilde{h} \circ \gamma_x \circ \tilde{h}^{-1} \end{aligned}$$

is a group isomorphism and if we change of lifting is a composition with a conjugation. $\square$

We want to show that universal orbifold covers exist and that an orbifold is the quotient of its universal cover by the action of the orbifold fundamental group. Let us define orbifold actions.

Definition 3.4. Let X be an orbifold and let G be a group. A *right orbifold action* of G on X is an antihomomorphism of groups from G to the group of orbifold diffeomorphisms of X . If φ_g is the image of $g \in G$ and $x \in |X|$, we have that $x \cdot g = \varphi_g(x)$. The action is properly discontinuous if the induced action on $|X|$ is.

Example 3.5. Let $X = \mathbb{S}_{(2)}^2$; we identify $\mathbb{S}^2$ with $\mathbb{C} \cup \{\infty\}$ and 0 is the singular point. The orbifold chart over 0 is the map $\Psi : \mathbb{C} \rightarrow \mathbb{C}$ given by $\Psi(z) := z^2$. We are going to consider two actions of μ_2 . Let us denote by α, β the images of -1 by the two actions, and by $\tilde{\alpha}, \tilde{\beta}$ their liftings:

$$\begin{array}{ccc} z & \xrightarrow{\quad} & iz \\ \left(\begin{array}{ccc} \mathbb{C} & \xrightarrow{\tilde{\alpha}} & \mathbb{C} \\ \downarrow \Psi & & \downarrow \Psi \\ \mathbb{C} & \xrightarrow{\alpha} & \mathbb{C} \end{array} \right) & & \left(\begin{array}{ccc} \mathbb{C} & \xrightarrow{\tilde{\beta}} & \mathbb{C} \\ \downarrow \Psi & & \downarrow \Psi \\ \mathbb{C} & \xrightarrow{\beta} & \mathbb{C} \end{array} \right) \\ z^2 & \xrightarrow{\quad} & -z^2 \end{array}$$

If we consider an orbifold structure on $|X|/\mu_2$ induced by the orbifold structure of X and the action of the group μ_2 , the natural orbifold chart would be the composition of Ψ and the quotient of $\mathbb{C} \subset |X|$ by the action of μ_2 . In the first case the isotropy group at 0 is μ_4 and in the second case it is $\mu_2 \times \mu_2$.

Remark 3.6. Let us consider a faithful properly discontinuous orbifold action of a group G on an orbifold X . Given $x \in |X|$ we have several finite groups: Γ_x is the isotropy group in X , G_x is the isotropy group of the action of G on $|X|$. Finally, let $\Gamma_{G,x}$ be the *group of liftings* of the action. Let us describe it.

It is possible to fix a chart $(\tilde{U}_x, U_x, \Gamma_x, \Psi_x)$, $x \in U_x$, such that for all $g \in G_x$ we have that $U_x \cdot g = U_x$. Then, $\Gamma_{G,x}$ is the group of all liftings in $\tilde{U}_x$ for all the elements of G . There is a short exact sequence

$$1 \longrightarrow \Gamma_x \longrightarrow \Gamma_{G,x} \longrightarrow G_x \longrightarrow 1.$$

In the previous example $\Gamma_x \cong \mu_2 \cong G_x$; we obtain as $\Gamma_{G,x}$ the two possible extensions for the two actions.

Definition 3.7. For a faithful properly discontinuous orbifold action of a group G on an orbifold X , the *associated orbifold* X_G is the orbifold such that $|X_G| = |X|/G$ and the charts are defined as in the previous remark.

Theorem 3.8. *Let X be a connected orbifold, and let $x \in |X|$ be a regular point.*

- (C1) *There exists a universal orbifold cover $p : \tilde{X} \rightarrow X$. Let us fix $\tilde{x} \in p^{-1}(x)$.*
- (C2) *Let $q : \hat{X} \rightarrow X$ is an orbifold cover and let $\hat{x} \in q^{-1}(x)$. Then there is a unique orbifold cover $h : \tilde{X} \rightarrow \hat{X}$ such that $p = q \circ h$ and $h(\tilde{x}) = \hat{x}$. In particular, the universal cover is unique up to diffeomorphism.*
- (C3) *The deck group G of p acts discrete and faithfully on $\tilde{X}$ and there is an orbifold diffeomorphism $\Phi : \tilde{X}_G \rightarrow X$ such that $|\Phi|$ is natural quotient map $|\tilde{X}|/G \rightarrow |X|$.*

Sketch of the proof. The universal abelian cover can be constructed as follows. Let

$$\Sigma := \{ \hat{\pi} : (\hat{X}, \hat{x}) \rightarrow (X, x) \mid \hat{\pi} \text{ is an orbifold cover} \}$$

be the set of pointed orbifold covers, ordered by factorization of mappings. A universal cover is a maximal element of Σ . Zorn's lemma provides such a maximal element. $\square$

Definition 3.9 (Cover-based orbifold fundamental group). The *orbifold fundamental group* of X is the deck group of its universal cover.

Both fundamental groups coincide and the main properties of usual covers and fundamental groups hold for their orbifold versions, including monodromy.

Properties 3.10. Let $p : \hat{X} \rightarrow X$ be an orbifold cover. Fix $\hat{x} \in \hat{X}$, denote $x := p(\hat{x})$, and assume that both points are regular in their orbifolds. Let $K := \pi_1^{\text{orb}}(\hat{X}; \hat{x})$ and $G := \pi_1^{\text{orb}}(X; x)$.

- (OC1) Orbifold paths and homotopies can be lifted.
- (OC2) The map p_* is injective. As in the classical case, we identify K and $p_*(K)$.
- (OC3) The cover is Galois (regular) if and only if $K \trianglelefteq G$; in that case if $H := G/K$, then there is a faithful discrete orbifold action of H on $\hat{X}$ such that X can be identified with the orbifold determined by the action of H on $\hat{X}$.
- (OC4) For any subgroup of G there is an orbifold cover such that the image of the orbifold fundamental group of the domain of the cover is that subgroup.
- (OC5) As a consequence for any epimorphism $\rho : G \rightarrow L$, L group, there is a Galois cover associated to $\ker \rho$. We will denote this cover as $p_\rho : X_\rho \rightarrow X$.

There is a particular class of cover maps which are interesting, where the total space is a manifold. There is an easy way to characterize those covers, at least when they are of Galois type. Let x be a regular point of an orbifold X and let $y \in |X|$. Let $(\tilde{U}_y, U_y, \Gamma_y, \Psi_y)$, $y \in U_y$ be a chart such that Γ_y is the isotropy group of y . Let us consider the orbifold $(\tilde{U}_y)_{\Gamma_y}$ and let y' be a regular point therein. We fix arbitrarily an orbifold path in X joining x and y' avoiding the singular part. Then $i_x : \pi_1^{\text{orb}}((\tilde{U}_y)_{\Gamma_y}; y') \rightarrow \pi_1^{\text{orb}}(X; x)$ is the morphism induced by $(\tilde{U}_y)_{\Gamma_y} \hookrightarrow X$ and the previous path. It is well defined up to conjugation.

Proposition 3.11. Let X be an orbifold and let $x \in X$ be a regular point. Let $\rho : \pi_1^{\text{orb}}(X; x) \rightarrow G$ be a group epimorphism. Then X_ρ is a manifold if and only if $\forall y \in X$ the composition $\rho \circ i_y$ is injective.

In the first series of talks orientable and non-orientable (connected) orbifolds were defined. As in the manifold case, there is a morphism group in μ_2 determining the orientability of an orbifold and in the non-orientable case define a double

(connected) orientable orbifold cover. We call this morphism the *orientation* morphism. Again as in the manifold case the orientable morphism passes to the covers. Namely, if $p : Y \rightarrow X$ is an orbifold cover and $\rho : \pi_1(X) \rightarrow \mu_2$ is the orientation morphism for X , then $\rho \circ p_*$ is the orientation morphism for Y .

Proposition 3.12. *Let X be a connected orbifold, x a regular point. There is a morphism $\rho : \pi_1^{\text{orb}}(X; x) \rightarrow \mu_2$ such that X is orientable if and only if ρ is trivial. If X is non-orientable, the orbifold cover p_ρ is the orientable orbifold double cover.*

For any Alice path γ , $\rho(\gamma) = -1$; hence, if there are Alice paths in $\pi_1^{\text{orb}}(X; x)$, then X is non-orientable.

Let us consider the orbifold P_n of Example 2.8. We have considered a double cover which eliminates the mirror singularities but it has n double conic singularities. If n is even, another double cover allows to recover a smooth surface, namely a hyperelliptic surface of genus $\frac{n-2}{2}$.

Since there are points with isotropy group $\mu_2 \times \mu_2$, no double cover of P_n is a manifold. Let us check how to obtain $\mu_2 \times \mu_2$ -Galois cover such that the total space is a manifold.

Let us recall the presentation of $\pi_1^{\text{orb}}(P_n)$ in (2.1). Each generator is of order 2 associated to an edge, and two generators commute if and only if the associated edges share a vertex. Consider three *colors* which we identify to $(1, 0), (0, 1), (1, 1) \in \mathbb{Z}/2 \times \mathbb{Z}/2$. Any coloring of the edges of the boundary of P_n , such that two adjacent edges have distinct colours, determine an epimorphism ρ on $\mathbb{Z}/2 \times \mathbb{Z}/2$, and the associated cover is a smooth surface. There are two types of singular points, with isotropy groups μ_2 and $\mu_2 \times \mu_2$. If x is of the first type, it corresponds to an edge and the image of i_x is the generator of this edge and the condition of Proposition 3.11 is clearly satisfied. If x is of the second type, it is a vertex, and the coloring hypothesis implies again the condition of Proposition 3.11. Hence, the cover is a manifold. The hyperelliptic cover corresponds to an alternating coloring.

Are they orientable? If the coloring has three colors, then, we have that the product of three generators is in $\ker \rho$, in particular it belongs to $\pi_1^{\text{orb}}(X_\rho)$, but its image by the orientation morphism is -1 , then, it is not orientable. Let us see how to obtain orientable covers.

Proposition 3.13. *Let $\rho : \pi_1^{\text{orb}}(P_n; O) \rightarrow (\mathbb{Z}/2)^k$, $k \leq n$, be a morphism which sends each γ_j to a basis element of $(\mathbb{Z}/2)^k$ and such that two consecutive generators are sent to different generators. Then, X_ρ is an orientable manifold.*

Proof. As in the previous discussion, the fact that the local maps are injective for all singular point guarantees that the cover is a manifold. Let $g \in \pi_1^{\text{orb}}(X_\rho)$; as a subgroup of $\pi_1^{\text{orb}}(X)$ it is $\ker \rho$. An element of $\ker \rho$ must be expressed as the product of an even number of generators, and then it is also in the kernel of the orientation morphism. Then X_ρ is orientable. $\square$

In particular, let $\tau : \pi_1^{\text{orb}}(P_n; O) \rightarrow (\mathbb{Z}/2)^n$ be the morphism sending each γ_i to a canonical basis element of $(\mathbb{Z}/2)^n$. This morphism determine an orbifold cover whose total space is an orientable smooth surface. The orbifold Euler characteristic of P_n equals

$$\frac{n}{4} - \frac{n}{2} + 1 = \frac{4-n}{4}.$$

Then, the Euler characteristic of this orbifold cover is multiplied by 2^n . Since the cover is compact and orientable, its genus is $g_n = 1 + 2^{n-3}(n-4)$, which appears as intersection of $n-2$ coaxial quadrics in $\mathbb{R}^n$, for $n \geq 2$. For more detailed explanations, see [7] and the references therein.

Remark 3.14. Note that higher homotopy groups have also an orbifold version using orbifold maps from spheres. As in the classical case for an orbifold S , $\pi_m^{\text{orb}}(S; x)$, $m \geq 2$, coincides with the corresponding group for any cover, in particular for the universal cover. As a consequence, for 2-orbifolds S , we have the following:

- ($\pi_m^{\text{orb}}1$) If $|S|$ has positive genus, $\pi_m^{\text{orb}}(S; x)$ is trivial.
- ($\pi_m^{\text{orb}}2$) If $|S|$ is a sphere, and there are at least 4 singular points, then $\pi_m^{\text{orb}}(S; x)$ is trivial.
- ($\pi_m^{\text{orb}}3$) If $S = \mathbb{S}_{(p,q,r)}^2$, $1 < p \leq q \leq r$, then $\pi_m^{\text{orb}}(S; x) \cong \pi_m(\mathbb{S}^2)$, $m \geq 2$, if and only if we are in the cases of (1.1), and trivial in the other cases (elliptic and hyperbolic).
- ($\pi_m^{\text{orb}}4$) If $S = \mathbb{S}^2$ or $\mathbb{S}_{n,n}^2$, then $\pi_m^{\text{orb}}(S; x) \cong \pi_m(\mathbb{S}^2)$, $m \geq 2$.
- ($\pi_m^{\text{orb}}5$) If $S = \mathbb{S}_{n,m}^2$, $1 \leq n < m$ we will compute the higher homotopy groups in Theorem 4.2. A candidate to be a non-trivial element of this group is given by the orbifold map $F : \mathbb{S}^2 \rightarrow S$ such that $|F|$ is the $\text{lcm}(n, m)$ -ramified cover of the sphere at two points. Let us identify $|S|, \mathbb{S}^2$ and $\mathbb{C} \cup \{\infty\}$, such that the 0 is the singular point of type n and ∞ is the singular point

of type m . With these identifications, $|F|(z) = z^{\text{lcm}(n,m)}$. We can take an orbifold chart $(\mathring{\mathbb{D}}, \mathring{\mathbb{D}}, \mu_n, \Psi)$, where $\Psi : \mathring{\mathbb{D}} \rightarrow \mathring{\mathbb{D}}$ is the ramified cover $\Psi(z) := z^n$. It is clear that it satisfies the orbifold chart properties. We have:

$$\begin{array}{ccc} & & \mathring{\mathbb{D}} \\ & \nearrow \tilde{F} & \downarrow \Psi \\ \mathbb{S}^2 \supset \mathring{\mathbb{D}} & \xrightarrow{|F|} & \mathring{\mathbb{D}} \subset |S| \end{array}$$

where $\tilde{F}(z) = z^{\frac{\text{lcm}(n,m)}{n}}$. Something similar goes for the other singular point.

4. SEIFERT FIBRATIONS AND LOCALLY TRIVIAL ORBIFOLD FIBRE BUNDLES

Seifert fibrations provide in a natural way the notion of orbifold bundles. The main reference is the original paper of Seifert [8], with an English translation in [9].

Definition 4.1. A *locally trivial orbifold fibre bundle* is a differentiable orbifold map $\Pi : E \rightarrow B$, such that for any $b \in |B|$, there are:

- (FB1) an orbifold chart $(\tilde{U}_b, U_b, \Gamma_b, \Phi_b)$, $b \in U_b$;
- (FB2) a manifold F ;
- (FB3) a continuous map $\Psi_b : \tilde{U}_b \times F \rightarrow |\Pi|^{-1}(U_b)$;
- (FB4) a discrete action of Γ_b on $\tilde{U}_b \times F$ such that the following Γ_b -equivariant diagram commutes

$$\begin{array}{ccccc} & & \tilde{U}_b \times F & \xrightarrow{\text{pr}_1} & \tilde{U}_b \\ & \swarrow & \downarrow \Psi_b & & \downarrow \Phi_b \\ (\tilde{U}_b \times F)/\Gamma_b & \xrightarrow{\cong} & |\Pi|^{-1}(U_b) & \xrightarrow{|\Pi|} & U_b; \end{array}$$

- (FB5) the orbifold structure over $|\Pi|^{-1}(U_b)$ is given by $(\tilde{U}_b \times F)_{\Gamma_b}$.

The long exact sequence of homotopy works for orbifold fibre bundles.

Theorem 4.2. Let $\Pi : E \rightarrow B$ an orbifold fibre bundle. Let $x \in E$ be a regular point of x in a regular fiber F (general fibers are connected), and let $y := \Pi(x)$. Then, there is a long exact sequence:

$$\dots \rightarrow \pi_2^{\text{orb}}(E; x) \rightarrow \pi_2^{\text{orb}}(B; y) \rightarrow \pi_1(F; x) \rightarrow \pi_1^{\text{orb}}(E; x) \rightarrow \pi_1^{\text{orb}}(B; y) \rightarrow 1.$$

Example 4.3. Let $\mathbb{V} := \mathbb{D} \times \mathbb{S}^1$. The first projection $\text{pr}_1 : \mathbb{V} \rightarrow \mathbb{D}$ is a trivial fiber bundle with fiber $\mathbb{S}^1$. Let us consider two actions of μ_p on $\mathbb{V}$ (free) and $\mathbb{D}$ (non-free) for some $p, q \in \mathbb{N}$ with $\gcd(p, q) = 1$:

$$(z, \lambda) \cdot \zeta := (z \cdot \zeta^{-q}, \lambda \cdot \zeta), \quad z \cdot \zeta := \zeta \cdot z.$$

Their quotients have a natural orbifold structure; since the actions are free, they are actually manifolds. We denote $\mathbb{V}_{p,q} := \mathbb{V}_{\mu_p}$ which is a solid torus. Actually, we have the following diagram where the horizontal arrow is a homeomorphism:

$$(4.1) \quad \begin{array}{ccccc} & & (z, \lambda) & & \\ & \swarrow & & \searrow & \\ & & \mathbb{V} & & \\ & \swarrow & & \searrow & \\ (z\lambda^q, \lambda^p) & \mathbb{V} & \xrightarrow{\cong} & \mathbb{V}_{p,q} & \end{array}$$

Sometimes it is easier to see $\mathbb{V}_{p,q}$ as the quotient of a cylinder:

$$\begin{aligned} C_{p,q} &:= \frac{\mathbb{D} \times [0,1]}{(z,1) \sim (\exp \frac{2i\pi q}{p} \cdot z, 0)} \longrightarrow \mathbb{V}_{p,q} \\ [(z, t)] &\longmapsto [(z, \exp 2i\pi p t)] \end{aligned}$$

There is a natural orbifold fibre bundle $\Phi_{p,q} : \mathbb{V}_{p,q} \rightarrow \mathbb{D}_{(p)}$ associated to

$$\begin{array}{ccc} \mathbb{D} \times \mathbb{S}^1 \equiv \mathbb{V} & \longrightarrow & \mathbb{D} \\ \downarrow & & \downarrow \\ \mathbb{V}_{p,q} & \xrightarrow{|\Phi_{p,q}|} & \mathbb{D}/\mu_p \\ [(z, \lambda)] & \longmapsto & [z]. \end{array}$$

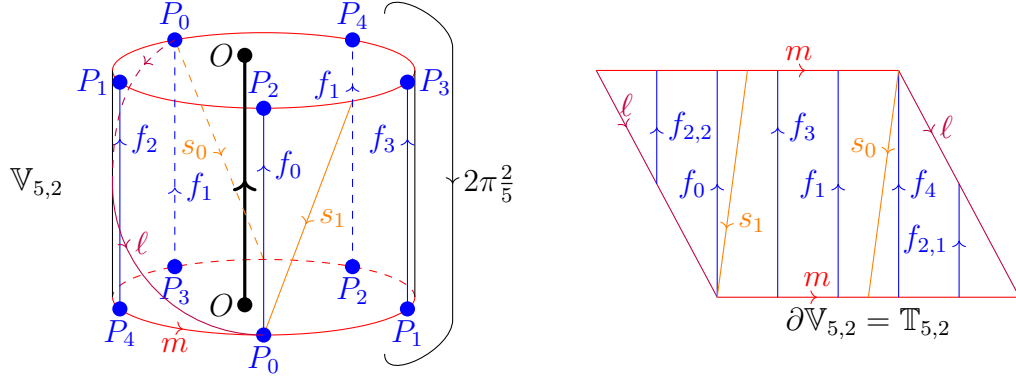
In terms of the homeomorphism of (4.1) this map is $(z, \lambda) \mapsto z^p \lambda^{-q}$. In terms of $C_{p,q}$, the map $|\Phi_{p,q}|$ can be seen as

$$\begin{aligned} \mathbb{V}_{p,q} \cong C_{p,q} &\xrightarrow{|\Phi_{p,q}|} \mathbb{D}/\mu_p \xrightarrow{\cong} \mathbb{D} \\ [(z, t)] &\longmapsto [z] \longmapsto z^p. \end{aligned}$$

This is the canonical example of a Seifert fibration.

Let us see some special 1-cycles in $\mathbb{V}_{p,q}$ and $\mathbb{T}_{p,q} := \partial \mathbb{V}_{p,q}$:

$$\begin{array}{lll} [0, 1] \xrightarrow{F_0} C_{p,q} & [0, 1] \xrightarrow{m} C_{p,q} & [0, 1] \xrightarrow{f} \mathbb{V}_{p,q} \\ t \longmapsto [(0, t)] & t \longmapsto [(\exp 2i\pi t, 0)] & t \longmapsto [(1, \exp 2i\pi p t)]. \end{array}$$

Figure 15. Solid torus $\mathbb{V}_{p,q}$, $p = 5$, $q = 2$.

The first one is the core F_0 of the solid torus and it generates $H_1(\mathbb{V}_{p,q}; \mathbb{Z}) \cong \mathbb{Z}$. The second one is a *meridian* m , homologically, a generator of the kernel of the map $i_* : H_1(\mathbb{T}_{p,q}; \mathbb{Z}) \rightarrow H_1(\mathbb{V}_{p,q}; \mathbb{Z})$ induced by the inclusion $\mathbb{T}_{p,q} \hookrightarrow \mathbb{V}_{p,q}$; this property characterizes this cycle up to sign. The third one is a *fiber* f of $\Phi_{p,q}$ and it is a well-defined cycle up to orientation. In Figure 15 is represented by several straight vertical segments which glue to produce a simple closed curve. Note that the image of f is $\Phi_{p,q}^{-1}([1])$ and $i_*(f) = p \cdot F_0$.

We are going to define a couple of cycles which are not canonical any more. Let us fix a cycle ℓ such that m, ℓ form a basis of $H_1(\mathbb{T}_{p,q}; \mathbb{Z})$; for example we can choose

$$\begin{aligned} [0, 1] &\xrightarrow{\ell} C_{p,q} \\ t &\longmapsto \left[\left(\exp \frac{2i\pi q t}{p}, 1 - t \right) \right]. \end{aligned}$$

in the model of the cylinder $C_{p,q}$ with identifications, we choose $\left(\exp \frac{2i\pi q t}{p}, 1 - t \right)$, as in Figure 15. Note that there is an infinite family of cycles that complete a basis with m , namely $\pm \ell + k \cdot m$, $k \in \mathbb{Z}$. For this particular choice of ℓ , it is not hard to check that

$$f = q \cdot m - p \cdot \ell.$$

Using Bézout identity consider $u, v \in \mathbb{Z}$ such that $p \cdot u + q \cdot v + 1 = 0$. As a consequence, if $s = u \cdot m + v \cdot \ell$, then s, f is another basis of $H_1(\mathbb{T}_{p,q}; \mathbb{Z})$. Actually it is also the case for $\pm s + k \cdot \ell$, for $k \in \mathbb{Z}$; see Figure 15 for a particular choice of s .

The important property of s is that there are simple closed curves representing s which are sections of the fibration $\Phi_{p,q}$. Let us see the transition matrices:

$$\begin{pmatrix} s & f \end{pmatrix} = \begin{pmatrix} m & \ell \end{pmatrix} \begin{pmatrix} u & q \\ v & -p \end{pmatrix} \implies \begin{pmatrix} m & \ell \end{pmatrix} = \begin{pmatrix} s & f \end{pmatrix} \begin{pmatrix} -p & -q \\ -v & u \end{pmatrix}.$$

Since $m + p \cdot s + v \cdot f = 0$ in $\mathbb{T}_{p,q}$, we deduce that $p \cdot s + v \cdot f = 0$ in $\mathbb{V}_{p,q}$.

For simplicity we are going to restrict our attention to oriented Seifert fibrations over oriented orbifolds.

Definition 4.4. An *oriented Seifert fibration* is an orbifold fibre bundle $\pi : M \rightarrow \Sigma$ where M is an oriented 3-manifold and Σ is an oriented 2-orbifold. Following the convention Base-Fibre, the orientations on M and Σ induce an orientation on the fibers.

Example 4.5. Let us view $\mathbb{S}^3$ as the boundary of the unit polydisk in $\mathbb{C}^2$; it is not a smooth manifold but it is easily smoothable. Let $p, q \geq 1$ and coprime. The map

$$\begin{aligned} \mathbb{S}^3 &\xrightarrow{\Psi_{p,q}} \mathbb{S}_{(p,q)}^2 \\ (x, y) &\longmapsto [x^p : y^q] \end{aligned}$$

is a Seifert fibration. Let $\mathbb{V}_x := \{(x, y) \in \mathbb{C}^2 \mid |x| \leq 1, |y| = 1\}$, which is a solid torus. The cycles m, ℓ, f , as in Example 4.3, are parametrized by

$$\begin{aligned} [0, 1] &\xrightarrow{m} \partial \mathbb{V}_x & [0, 1] &\xrightarrow{\ell} \partial \mathbb{V}_x & [0, 1] &\xrightarrow{f} \partial \mathbb{V}_x \\ t &\longmapsto (e^{2i\pi t}, 1) & t &\longmapsto (1, e^{-2i\pi t}) & t &\longmapsto (e^{2i\pi t q}, e^{2i\pi t p}). \end{aligned}$$

It is easily checked that $f = q \cdot m - p \cdot \ell$, and the restriction of $\Psi_{p,q}$ looks like $\Phi_{p,q}$. In the same way the restriction to $\mathbb{V}_y := \{(x, y) \in \mathbb{C}^2 \mid |y| \leq 1, |x| = 1\}$ looks like $\Phi_{q,p}$.

Example 4.6. Let us consider the action of μ_p to $\mathbb{S}^3$ where $(x, y) \cdot \zeta := (\zeta \cdot x, \zeta^q \cdot y)$, where $\gcd(p, q) = 1$. This is a free action, and its quotient is the lens space $L(p, q)$. Let (a, b) a pair of coprime integers such that $a \equiv b \cdot q \pmod{p}$. The following diagram commutes for $\zeta \in \mu_p$:

$$\begin{array}{ccc} \mathbb{S}^3 & \xrightarrow{\cdot \zeta} & \mathbb{S}^3 \\ \Psi_{a,b} \searrow & & \swarrow \Psi_{a,b} \\ & \mathbb{S}_{(a,b)}^2 & \end{array}$$

and the map pass to the quotient as a Seifert map.

Remark 4.7. A Seifert fibration provides a long exact sequence of homotopy:

$$1 \rightarrow \pi_2(M; x) \rightarrow \pi_2^{\text{orb}}(S; y) \rightarrow \mathbb{Z} \rightarrow \pi_1(M; x) \rightarrow \pi_1^{\text{orb}}(S; y) \rightarrow 1,$$

and $\pi_m(M; x) \cong \pi_m^{\text{orb}}(S; y)$ for $m \geq 3$. For the bad orbifolds $S = \mathbb{S}_{(n)}^2$, $n > 1$, or $\mathbb{S}_{(n,m)}^2$, $n < m$, we have Seifert fibrations with total space $\mathbb{S}^3$. In particular, we that $\pi_2^{\text{orb}}(S) \cong \mathbb{Z}$, and for higher homotopy groups we recover the groups of $\mathbb{S}^3$.

Properties 4.8. Let us recall some properties of the oriented fiber bundles over oriented surfaces. It is well-known that when the base has non-empty boundary, these fibre bundles are trivial, but we can be more specific.

- (a) Let $\pi : M \rightarrow S$ be an oriented $\mathbb{S}^1$ -fibre bundle over an oriented 2-manifold. If $\partial S \neq \emptyset$, let $\{C_0, C_1, \dots, C_n\}$, $n \geq 0$, be the boundary connected components, with the natural orientation. Let $T_i := \pi^{-1}(C_i)$, an oriented torus for each $i \in \{0, 1, \dots, n\}$. Fix sections $s_i \subset T_i$, $i = 1, \dots, n$, such that $\pi(s_i) = C_i$ (taking the orientation into account). Then, up to isotopy there is exactly one section $s : S \rightarrow M$ such that $s(S) \cap T_i = s_i$, $i > 0$. Let $s_0 := s(S) \cap T_0$; as a cycle in $H_1(T_0; \mathbb{Z})$ it is completely determined by $s_1, \dots, s_n$.
- (b) Actually, the class of s_0 in T_0 is completely determined by the orientation of C_0 and by the fact that $s_0 + s_1 + \dots + s_n \in \mathbb{Z}\langle f \rangle$, where f is the class of a fiber and the sum is taken in $H_1(M; \mathbb{Z})$.
- (c) Let $\pi : M \rightarrow S$ be an oriented $\mathbb{S}^1$ -fibre bundle over an oriented 2-manifold S , where $\{C_1, \dots, C_n\}$ are the connected components of the boundary, $n \geq 0$. Fix oriented sections $s_i \subset T_i$, $i = 1, \dots, n$, such that $\pi(s_i) = C_i$. Then, the *Euler number* of π with respect to $(s_1, \dots, s_n)$ is a number $e \in \mathbb{Z}$ obtained as follows. Take a closed disk $\mathbb{D}_0 \subset \mathring{S}$, and let $S^\vee := S \setminus \mathring{\mathbb{D}}_0$; let $C_0 = \partial \mathbb{D}_0$, where its orientation is taken as part of $\partial S^\vee$. Apply (a) to $S^\vee$ and consider the component s_0 . Following (b), the Euler number e is the integer satisfying $s_0 + s_1 + \dots + s_n = e \cdot f$.
- (d) With a suitable choice of paths, if g is the genus of S we have that

$$\pi_1(M) \cong \left\langle \begin{array}{c} \sigma_i, a_i, b_i, \varphi \\ 1 \leq i \leq n \quad 1 \leq i \leq g \end{array} \middle| \begin{array}{c} [a_i, \varphi] = [b_i, \varphi] = 1, [\sigma_i, \varphi] = 1, \\ 1 \leq i \leq n \end{array} \prod_{i=1}^g [a_i, b_i] \cdot \prod_{i=1}^n \sigma_i = \varphi^e \right\rangle.$$

- (e) The previous definition applies to the case where $\partial S = \emptyset$.

Examples 4.5 and 4.6 seems to indicate that a manifold can admit many Seifert fibrations. One of the results of Seifert is that it is not the case. Actually, in the oriented case, these examples are basically the only ones where there is no unicity.

The strategy of Seifert passed to use the Euler number and the possibility to choose the section s of Example 4.3, such that the value v is in $(0, p)$. We summarize this in the following proposition, which follows from Seifert-van Kampen theorem.

Proposition 4.9. *Let $\pi : M \rightarrow S$, g the genus of S , a Seifert fibration having r exceptional fibers $\mathbb{V}_{p_i, q_i}$. Let us choose the sections s_i such that the meridians m_i are expressed as $-(p_i \cdot s_i + v_i \cdot f)$. Let e be the Euler number of the fibration outside the exceptional fibers with respect to the sections s_i . Then,*

$$\pi_1(M) \cong \left\langle \begin{array}{c} \sigma_i, a_i, b_i, \varphi \\ 1 \leq i \leq r \quad 1 \leq i \leq g \end{array} \middle| \varphi \text{ central}, \prod_{i=1}^g [a_i, b_i] \cdot \prod_{i=1}^n \sigma_i = \varphi^e, \sigma_i^{p_i} \cdot \varphi^{v_i} = 1 \right\rangle.$$

In dimension 4 we are going to study locally trivial orbifold fibre bundles over 2-orbifolds having 2-tori as fibers. Let us start with the local models as the fibrations over $\mathbb{D}_{(p)}$.

Example 4.10. The torus is a compact orientable topological surface that carries a family of structures as analytic complex curve. These analytic structures are of the form $\mathbb{T}_\Lambda := \mathbb{C}/\Lambda$ where Λ is a lattice, i.e. a rank 2 free abelian subgroup of $\mathbb{C}$ (for the sum), which generates $\mathbb{C}$ as $\mathbb{R}$ -vector space. These structures are called *elliptic curves*.

It is easily seen that $\mathbb{T}_\Lambda$ is isomorphic to $\mathbb{T}_{w \cdot \Lambda}$, if $w \neq 0$. As a consequence, it is enough to study the lattices of the form $\Lambda_\lambda := \mathbb{Z}\langle 1, \lambda \rangle$, where

$$\lambda \in \mathbb{H} := \{t \in \mathbb{C} \mid \Im t > 0\}.$$

We will denote $\mathbb{T}_\lambda := \mathbb{T}_{\Lambda_\lambda}$. It is well-known that two lattices $\mathbb{T}_{\lambda_1}, \mathbb{T}_{\lambda_2}$ are isomorphic if and only if they are related by a modular transformation, i.e.,

$$\lambda_2 = \frac{a\lambda_1 + b}{c\lambda_1 + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2; \mathbb{Z}).$$

The notion of elliptic fibrations is a very useful notion in the study of complex surfaces; they are holomorphic maps between a (compact) complex surface and a (compact) complex curve (a Riemann surface) such that the generic fiber is an elliptic curve. Kodaira [5] classified the singular fibers and some of them can be interpreted from an orbifold point of view.

Definition 4.11. A *complex torus orbifold fibre bundle* is defined as an orbifold map $\pi : X \rightarrow S$ where X, S are compact complex orbifolds of dimension 1, 2, respectively, and such that π is a locally trivial orbifold fibre bundle in the differentiable category. It is said to be *isotrivial* if it is locally trivial in the holomorphic category.

Applying the long exact sequence of homotopy we have:

$$1 \rightarrow \pi_2^{\text{orb}}(X; x) \rightarrow \pi_2^{\text{orb}}(S; y) \rightarrow \mathbb{Z}^2 \rightarrow \pi_1^{\text{orb}}(X; x) \rightarrow \pi_1^{\text{orb}}(S; y) \rightarrow 1,$$

and isomorphisms for the higher homotopy groups of X and S . Let us recall that if S is a compact orbifold then $\pi_2^{\text{orb}}(S; y)$ is either trivial or infinite cyclic.

Let us study locally these torus orbifold fibre bundles. We start describing the regular fibers over disks. We will consider neighbourhoods of points in complex curves and we identify them to closed disks $\mathbb{D} \subset \mathbb{C}$, and by *holomorphic functions* on $\mathbb{D}$ we mean the restrictions of holomorphic functions in a small open neighbourhood of $\mathbb{D}$.

Definition 4.12. Let $\tau : \mathbb{D} \rightarrow \mathbb{H}$ be holomorphic and let us consider an action of $\mathbb{Z}^2$ on $\mathbb{D} \times \mathbb{C}$ defined by:

$$(4.2) \quad (z, w) \cdot (1, 0) := (z, w + 1), \quad (z, w) \cdot (0, 1) := (z, w + \tau(z)).$$

The quotient $\mathbb{E}_\tau := (\mathbb{D} \times \mathbb{C})/\mathbb{Z}^2$ is called a τ -local model.

Note that for $z \in \mathbb{C}$, $\text{pr}_1^{-1}(z)$ is isomorphic to the elliptic curve $\mathbb{T}_{\tau(z)}$.

Definition 4.13. Let $\pi : X \rightarrow S$ be a complex torus orbifold fibre bundle, and let $p \in S$ be a regular point of S . We say that p is *differentially regular* (for short, regular) for p if there is a neighbourhood V of π , a holomorphic function $\tau : \mathbb{D} \rightarrow \mathbb{H}$ and isomorphisms which complete the following diagram:

$$\begin{array}{ccc} \pi^{-1}(V) & \xrightarrow{\cong} & \mathbb{E}_\tau \\ \downarrow \pi & & \downarrow \text{pr}_1 \\ V & \xrightarrow{\cong} & \mathbb{D} \end{array}$$

The point will be said to be *holomorphically regular* if the map τ is a constant.

As for the Seifert fibrations, the idea to construct *singular fibers* over orbifold points uses cyclic actions over $\mathbb{E}_\tau$ (for some special τ 's) such that the action on the second factor is by finite order automorphisms of the fibers.

Example 4.14. The translations define automorphisms for all elliptic curves $\mathbb{T}_\Lambda$. They have several types of finite order automorphisms:

- (A1) For $w \in \frac{1}{d}\Lambda$ (d minimal for such condition), $\tau_w : \mathbb{T}_\Lambda \rightarrow \mathbb{T}_\Lambda$, $\tau([z]) = [z + w]$.
- (A2) $\pi_{-1} : \mathbb{T}_\Lambda \rightarrow \mathbb{T}_\Lambda$, $\pi_{-1}([z]) = [-z]$.
- (A3) $\pi_i : \mathbb{T}_i \rightarrow \mathbb{T}_i$, $\pi_i([z]) = [i \cdot z]$.
- (A4) $\pi_{\pm\zeta} : \mathbb{T}_\zeta \rightarrow \mathbb{T}_\zeta$, $\pi_{\pm\zeta}([z]) = [\pm\zeta \cdot z]$, for $\zeta = \exp \frac{2i\pi}{3}$.

This is the complete list of finite order automorphisms up to conjugation and isomorphism. Actually, up to conjugation and isomorphism the automorphisms of (A1) are of the form $\tau_{\frac{1}{d}} : \mathbb{T}_\lambda \rightarrow \mathbb{T}_\lambda$, for some $\lambda \in \mathbb{H}$.

Definition 4.15. Let $\tau : \mathbb{D} \rightarrow \mathbb{H}$ be holomorphic and $p \in \mathbb{Z}_{>1}$. We define a (τ, p) -local model $\mathbb{E}_{\tau,p}$ as in Definition 4.12 replacing the second formula in (4.2) by

$$(4.3) \quad (z, w) \cdot (0, 1) := (z, w + \tau(z^p)).$$

This implies that the fibers over z and over $\zeta \cdot z$ are isomorphic as analytic curves if $\zeta \in \mu_p$.

Example 4.16. Fix τ and $p \in \mathbb{Z} > 1$ as above, and let q be coprime with p . Consider an action of $\mathbb{Z}/p$ on $\mathbb{E}_{\tau,p}$:

$$[(z, w)] \cdot [1] := \left[\left(z \cdot \exp \frac{-2i\pi q}{p}, w + \frac{1}{p} \right) \right],$$

In multiplicative language with μ_p :

$$[(z, w)] \cdot \zeta := \left[\left(z \cdot \zeta^{-q}, w + \frac{\log \zeta}{2i\pi} \right) \right].$$

The action is free and the quotient $\mathbb{E}_{\tau,p,q} = \mathbb{E}_{\tau,p}/\mu_p$ is an analytic manifold.

Proposition 4.17. *With the above notation, the map $\text{pr}_1 : \mathbb{E}_{\tau,p,q} \rightarrow \mathbb{D}_{(p)}$ is a complex torus orbifold fiber bundle defined by*

$$(4.4) \quad \begin{array}{ccc} \mathbb{E}_{\tau,p} & \xrightarrow{\text{pr}_1} & \mathbb{D} \\ \downarrow \Psi & & \downarrow \Phi \\ \mathbb{E}_{\tau,p,q} & \xrightarrow{|\text{pr}_1|} & \mathbb{D}_{(p)}. \end{array}$$

The fibers over $[z]$, $z \neq 0$, are isomorphic to $\mathbb{T}_\tau(z^p)$ and the restriction of Φ to $\{z\} \times \mathbb{T}_\tau(z^p)$ is an isomorphism onto the image.

The fiber over $[0]$ is isomorphic to $\mathbb{T}_\Lambda$ for $\Lambda = \left\langle \frac{1}{p}, \tau(0) \right\rangle$ and the restriction of Φ to $\{0\} \times \mathbb{T}_\tau(0)$ is an unramified $p : 1$ cover. This is why the central fiber is said to be multiple.

From the automorphisms of Example 4.14, the ones of (A1) are the only ones without fixed points. With the other finite order automorphisms we can obtain other complex torus orbifold fibre bundles.

Proposition 4.18. *Let $\tau : \mathbb{D} \rightarrow \mathbb{H}$ be a holomorphic map and let $\mathbb{E}_{\tau,p}$ be the quotient by the action of $\mathbb{Z}^2$ defined by (4.2) and (4.3). Let us consider the following actions of μ_p :*

- (S1) $p = 2$, $[(z, w)] \cdot (-1) := [(-z, -w)]$.
- (S2) $p = 3$, $\tau(0) = \exp \frac{2i\pi}{3} := \zeta$, $[(z, w)] \cdot \zeta := [(\exp \zeta^{\pm 1} \cdot z, \zeta \cdot w)]$.
- (S3) $p = 4$, $\tau(0) = i$, $[(z, w)] \cdot i := [(\pm i \cdot z, i \cdot w)]$.
- (S4) $p = 6$, $\tau(0) = -\exp \frac{2i\pi}{3} := \omega$, $[(z, w)] \cdot \omega := [(\exp \omega^{\pm 1} \cdot z, \zeta \cdot \omega)]$.

We denote by $\mathbb{E}_{\tau,p,\xi}$ the orbifold $(\mathbb{E}_{\tau,p})_{\mu_p}$ for

$$(p, \xi) = (2, -1), (3, \zeta^{\pm 1}), (4, \pm i), (6, \omega^{\pm 1}).$$

The maps $\text{pr}_1 : \mathbb{E}_{\tau,p,\xi} \rightarrow \mathbb{D}_{(p)}$ are complex torus orbifold fiber bundles defined by diagrams similar to (4.4). The fibers over $[z]$, $z \neq 0$, are isomorphic to $\mathbb{T}_\tau(z^p)$ and the restriction of Φ to $\{z\} \times \mathbb{T}_\tau(z^p)$ is an isomorphism onto the image. For the central fibers over $[0]$ we obtain orbifold isomorphic to $\mathbb{S}_{(2,2,2,2)}^2$ (S1), $\mathbb{S}_{(3,3,3)}^2$ (S2), $\mathbb{S}_{(2,4,4)}^2$ (S3), and $\mathbb{S}_{(2,3,6)}^2$ (S4). The singular points of the orbifolds are the unique singular points of the total spaces.

We can also mix both actions and obtain *multiple orbifold fibers*. In order to study the global complex torus orbifold fiber bundles, we can proceed as we did with Seifert fibrations. In order to compute the fundamental groups we used

implicitly that the inclusion $i : \mathbb{T}_{p,q} \hookrightarrow \mathbb{V}_{p,q}$ induced a surjection $i_* : \mathbb{Z}^2 \equiv \pi_1(\mathbb{T}_{p,q}) \rightarrow \pi_1^{\text{orb}}(\mathbb{V}_{p,q}) \equiv \mathbb{Z}$, and the key information from the fundamental group come from $\ker i_*$.

Something similar can be done for the above singular fibers. Let us denote the total orbifolds by $\mathbb{E}$, and we have the diagram

$$\begin{array}{ccccc} \partial\mathbb{E}_{\tau,p} & \xrightarrow{\tilde{i}} & \mathbb{E}_{\tau,p} & \xrightarrow{\text{pr}_1} & \mathbb{D} \\ \downarrow \Psi| & & \downarrow \Psi & & \downarrow \Phi \\ \partial\mathbb{E}_{\tau,p,q} & \xrightarrow{i} & \mathbb{E}_{\tau,p,q} & \xrightarrow{|\text{pr}_1|} & \mathbb{D}_{(p)}. \end{array}$$

The vertical maps are orbifold covers and $\Psi^{-1}(\partial\mathbb{E}_{\tau,p,q}) = \partial\mathbb{E}_{\tau,p}$ is connected, which implies by standard properties of orbifold covers that the map induced by i on orbifold fundamental groups is also surjective.

In the section about complex orbifolds we will relate these orbifold fibers with Kodaira's classification [5].

5. COMPLEX ORBIFOLDS

Complex orbifolds are standard orbifolds where we replace $\mathbb{R}^n$ by $\mathbb{C}^n$ and differentiable by holomorphic. In particular complex orbifolds are orientable (they may be non-orientable if we may allow also antiholomorphic maps). The oriented orbifolds (without boundary) in real dimension 2 can be seen as complex orbifolds in dimension 1; the analytic structure implies the existence of moduli spaces.

Example 5.1. Let us start with an alternative description of the simply-connected bad 2-orbifolds. Let $p_x, p_y \in \mathbb{Z}_{>0}$ be coprime integers; let us denote $\omega := (p_x, p_y)$. We consider the following action of $\mathbb{C}^*$ on $\mathbb{C}^2 \setminus \{0\}$:

$$t \cdot (x, y) := (t^{p_x} x, t^{p_y} y).$$

The closure of the orbits have equations $\alpha x^{p_y} = \beta y^{p_x}$ for some $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{0\}$. Let us denote by X the quotient and by $[x : y]_\omega$ the equivalence class of (x, y) . Following the description of the projective line we can consider the following *charts* if we denote by U_x the image in X of $x \neq 0$ and by U_y the image in X of $y \neq 0$ (open sets in the quotient topology):

$$\begin{array}{ccc} \mathbb{C}_x & \xrightarrow{\Psi_x} & U_x \\ y & \longmapsto & [1 : y]_\omega, \end{array} \quad \begin{array}{ccc} \mathbb{C}_y & \xrightarrow{\Psi_y} & U_y \\ x & \longmapsto & [x : 1]_\omega. \end{array}$$

These maps are continuous and surjective but they are not injective. If $\zeta \in \mu_{p_x}$, then $\Psi_x(\zeta^{p_y}t) = [1 : \zeta^{p_y}t]_\omega = [\zeta^{p_x} : \zeta^{p_y}t]_\omega = [1 : t]_\omega$; since p_y is coprime with p_x it is not difficult to check that $\Psi_x(t_1) = \Psi_x(t_2)$ if and only if $t_2 = \zeta t_1$ for some $\zeta \in \mu_{p_x}$. We can see $(\mathbb{C}_x, U_x, \mu_{p_x}, \Psi_x)$ as an orbifold chart, and something similar happens with $(\mathbb{C}_y, U_y, \mu_{p_y}, \Psi_y)$; moreover, taking suitable domains where roots are holomorphic, the charts are compatible and they define an orbifold structure on X which will be called the *weighted projective line of weights* $\omega = (p_x, p_y)$ and denoted by $\mathbb{P}_\omega^1 = \mathbb{P}_{(p_x, p_y)}^1$. The map

$$(5.1) \quad \begin{aligned} \mathbb{P}_\omega^1 &\longrightarrow \mathbb{P}^1 \\ [x : y]_\omega &\longmapsto [x^{p_y} : y^{p_x}] \end{aligned}$$

is a bijective orbifold map, but its inverse is not an orbifold map. Note that $\mathbb{P}_\omega^1$ is simply connected.

Example 5.2. Let $X := \mathbb{P}_{(m_1, \dots, m_r)}^1$ be an orbifold such that $\left| \mathbb{P}_{(m_1, \dots, m_r)}^1 \right| = \mathbb{P}^1$ and there are points $P_1, \dots, P_r$ such that the isotropy group of P_i is μ_{m_i} , $i = 1, \dots, r$. The orbifold structure comes from the maximal atlas containing the following orbifold charts:

- (a) If $P \neq P_i$, take a usual chart $\Psi_P : \mathring{\mathbb{D}} \rightarrow U_P$, such that $\Psi_P(0) = P$ and $P_1, \dots, P_r \notin U_P$. The orbifold chart is $(\mathring{\mathbb{D}}, U_P, \{1\}, \Psi_P)$.
- (b) If $P = P_i$, take a usual chart $\Psi_P : \mathring{\mathbb{D}} \rightarrow U_P$, such that $\Psi_P(0) = P_i$ and $P_1, \dots, P_{i-1}, P_{i+1}, \dots, P_r \notin U_P$. Let $\tilde{\Psi}_P : \mathring{\mathbb{D}} \rightarrow U_P$ defined by $\tilde{\Psi}_P(z) = \Psi_P(z^{m_i})$. The orbifold chart is $(\mathring{\mathbb{D}}, U_P, \mu_{m_i}, \tilde{\Psi}_P)$.

As in the previous case, it is not hard to check the compatibility of the charts and we obtain a complex orbifolds. If $r = 2$, $P_1 = [0 : 1]$, and $P_2 = [1 : 0]$ the map of (5.1) defines an orbifold isomorphism and the notations become consistent. Applying Seifert-van Kampen Theorem we obtain

$$\pi_1^{\text{orb}}(\mathbb{P}_{(m_1, \dots, m_r)}^1) = \langle \gamma_1, \dots, \gamma_r \mid \gamma_1^{m_1} = \dots = \gamma_r^{m_r} = \gamma_1 \cdot \dots \cdot \gamma_r = 1 \rangle.$$

Analogously for $\mathbb{C}_{(m_1, \dots, m_r)}$, its fundamental group is $\mu_{m_1} * \dots * \mu_{m_r}$. The same ideas work for any Riemann surface.

We are going to interpret the pencils of curves in a complex surface using orbifolds. A *pencil* of curves is a dominant rational map $S \dashrightarrow \mathbb{P}^1$ from a complex surface. The name of pencil comes from the following important case. If

$F_1, F_2 \in \mathbb{C}[x, y, z]_d$ are coprime homogeneous polynomials of degree d , then

$$(5.2) \quad \begin{array}{ccc} \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^1 \\ [x : y : z] & \longmapsto & [F_1(x, y, z) : F_2(x, y, z)] \end{array}$$

and the fibers are the curves of the form $\alpha F_1(x, y, z) = \beta F_2(x, y, z)$ with $(\alpha, \beta) \neq 0$; these curves form a pencil of degree d . When some elements of the pencil are not reduced, the pencil map can be interpreted as an orbifold map.

Example 5.3 ([10]). Zariski studied a curve $C_6 \subset \mathbb{P}^2$ with equation $f_2(x, y, z)^3 - f_3(x, y, z)^2 = 0$, where f_m , $m = 2, 3$, are *generic* polynomials of degree m , with C_m the curve of equation $f_m(x, y, z) = 0$. Let us consider

$$(5.3) \quad \begin{array}{ccc} \mathbb{P}^2 & \dashrightarrow^{\Phi} & \mathbb{P}^1 \\ [x : y : z] & \longmapsto & [f_2(x, y, z)^3 : f_3(x, y, z)^2]. \end{array}$$

The genericity condition implies that $\#B = 6$, where $B := C_2 \cap C_3$ and Φ is a surjective morphism on $\mathbb{P}^2 \setminus B$, with irreducible fibers. The fibers are the curves $C_{[\alpha:\beta]}$ with equation $\beta f_2(x, y, z)^3 - \alpha f_3(x, y, z)^2 = 0$, $[\alpha : \beta] \in \mathbb{P}^1$, and for generic values of (α, β) , B coincides with the singular locus of $C_{[\alpha:\beta]}$. We may assume that $[1 : 1]$ is a generic value and note that $C_6 = C_{[1:1]}$. Any singular curve C in $\mathbb{P}^2$ is the image of a Riemann surface by a map which is generically $1 : 1$, called its normalization. The normalization is connected if and only if C is irreducible. In particular for generic values $\Phi^{-1}([\alpha : \beta]) = C_{[\alpha:\beta]} \setminus B$ is connected. If φ is the forget map (bijective), we can define a continuous map $|\Phi_{(2,3)}| : \mathbb{P}^2 \setminus B \rightarrow |\mathbb{P}_{(2,3)}^1|$, and this map has liftings that guarantee the existence of $\Phi_{(2,3)}$. If $P \in \Phi^{-1}([1 : 0]) \subset C_3$ note that $P \notin C_2$. Let $\Psi : U \rightarrow \mathbb{P}^2$ be a *small* chart centered at P . The following diagram shows that the lift is possible at P and a similar argument works for $[0 : 1]$.

$$\begin{array}{ccccc} & & & \frac{f_3(\Psi(u, v))}{f_2(\Psi(u, v))^{\frac{3}{2}}} = w & \\ & \nearrow & & \downarrow & \nwarrow \\ \mathbb{P}^2 \setminus B & \xrightarrow{\Phi} & \mathbb{P}^1 & & \\ \uparrow \Phi_{(2,3)} & & \downarrow \varphi & & \\ \mathbb{P}^2 & & \mathbb{P}^1_{(2,3)} & & \\ \uparrow & & \downarrow & & \\ U & \xrightarrow{\quad} & |\mathbb{P}_{(2,3)}^1| & & \\ (u, v) & & [f_3(\Psi(u, v))^3 : f_3(\Psi(u, v))^2] = [1 : w^2] & & \end{array}$$

We can restrict this map $\Phi_1 : \mathbb{P}^2 \setminus C_6 \rightarrow \mathbb{P}_{(2,3)}^1 \setminus \{[1 : 1]\} \equiv \mathbb{C}_{(2,3)}$.

Let us consider a path loop in the target. It is not hard to see that it can be lifted to a path in $\mathbb{P}^2 \setminus C_6$, with end points in the same fiber. Since the fiber are connected, we deduce that

$$\pi_1(\mathbb{P}^2 \setminus C_6) \xrightarrow{\Phi_{1*}} \pi_1^{\text{orb}}(\mathbb{C}_{(2,3)}) \cong \mu_2 * \mu_3$$

is surjective. Zariski proved that these groups are actually isomorphic (actually that this map is an isomorphism) using long computations.

Example 5.4. In [10], it was also proved that the fundamental group of the complement of the deltoid is a non-abelian fundamental group of order 12. We are going to construct an epimorphism on this group onto the dihedral group of order 6.

One equation for the deltoid C_4 is

$$f_4(x, y, z) := x^2y^2 + y^2z^2 + z^2x^2 - 2xyz(x + y + z) = 0,$$

and its bitangent is $C_1 : f_1(x, y, z) := x + y + z = 0$. Let us consider the following conic and cubic for ζ a primitive cubic root of unity:

$$C_2 : f_2(x, y, z) := xy + \zeta^{-1}xz + \zeta yz = 0$$

$$C_3 : f_3(x, y, z) := x^2(y - z) + y^2(z - x) + z^2(x - y) + 3(\zeta - \zeta^{-1})xyz = 0.$$

We have $4f_2^3 + f_3^2 = f_4f_1^2$. Let $B := C_2 \cap C_3$, a finite set with four points, the three cusps of C_4 and one tangency point in C_1 . The map

$$\begin{aligned} \mathbb{P}^2 \setminus B &\xrightarrow{F} \mathbb{P}^1 \\ [x : y : z] &\longmapsto [f_2(x, y, z)^3 : f_3(x, y, z)^2], \end{aligned}$$

defines an orbifold map if we consider the target as $\mathbb{P}_{(2,3)}^1$, where the singular points of order 3, 2 are $[0 : 1], [1 : 0]$, respectively. Note that $F(C_4) = F(L) = [1 : -4]$. The important point is that the restriction of F to $\mathbb{P}^2 \setminus C_4$ defines an orbifold map to $\mathbb{P}_{(2,3,2)}^1$, with generic connected fiber. In particular, $\pi_1(\mathbb{P}^2 \setminus C_4)$ has a surjection onto the triangle group $(2, 2, 3)$, i.e., the dihedral group of 6 elements.

Note that a similar argument works for:

$$\begin{aligned} g_2(x, y, z) &:= xy \\ g_3(x, y, z) &:= (x - y)^2z - xy(x + y) \\ g_1(x, y, z) &:= x - y. \end{aligned}$$

Note that $4g_2^3 - g_3^2 + g_1^2 f_4 = 0$. We have at least 5 different orbifold maps onto $\mathbb{P}_{(2,3,2)}^1$ with generic connected fiber.

Remark 5.5. Actually we can construct primitive pencils in $\mathbb{P}^2$ (with generic irreducible fiber) with any number of non-reduced fibers.

Let us study now complex 2-orbifolds (which are real 4-orbifolds). In Example 1.5 we defined already an orbifold $\mathbb{C}_{\mu_n \times \mu_m}^2$ with underlying space $\mathbb{C}^2$. When $n = m$ we can extend this action to the projective plane and we have an action

$$\underbrace{\overline{\langle (\zeta, \zeta, \zeta) \mid \zeta \in \mu_n \rangle}}_{G_n \cong \mu_n^2} \times \mathbb{P}^2 \longrightarrow \mathbb{P}^2$$

where $[\zeta_x, \zeta_y, \zeta_z] \cdot [x : y : z] := [\zeta_x x : \zeta_y y : \zeta_z z]$. Note that $\mathbb{P}_{G_n}^2$ has $\mathbb{P}^2$ as underlying space, with points of isotropy groups which are trivial ($xyz \neq 0$), μ_n (one variable vanishes), or $\mu_n \times \mu_n$ (two variables vanish); the universal cover is $\Phi : \mathbb{P}^2 \rightarrow \mathbb{P}_{G_n}^2$ where $|\Phi|([x : y : z]) = [x^n : y^n : z^n]$. If we forget the orbifold structure it is a ramified cover, called a *Kummer cover*.

Let us study now some other covers where the underlying space maybe singular.

Example 5.6. Let us consider the action of μ_2 on $\mathbb{C}^2$ such that $(-1) \cdot (x, y) := (-x, -y)$. As an algebraic variety the quotient is singular and isomorphic to $C := \{(u, v, w) \in \mathbb{C}^3 \mid v^2 = uw\}$ via the map

$$\begin{aligned} \mathbb{C}^2 &\longrightarrow C \\ (x, y) &\longmapsto (x^2, xy, y^2). \end{aligned}$$

The boundary of a regular neighbourhood of $[(0, 0)]$ in $\mathbb{C}^2/\mu_2$ is the real projective space (or equivalently the lens space $L(2, 1)$). We denote the orbifold $\mathbb{C}_{\mu_2}^2$ as $\frac{1}{2}(1, 1)$. Sometimes we will identify the orbifold with C , since it is easier to deal with the 2 *orbifold coordinates* than with the three actual coordinates (subject to a relation).

Definition 5.7. Let $d \in \mathbb{Z}_{>0}$, and $(a, b) \in \mathbb{Z}^2$, where $\gcd(d, a, b) = 1$. The *cyclic orbifold* $(\frac{1}{d}; a, b)$ is the orbifold $\mathbb{C}_{\mu_d}^2$ defined by the action $\zeta \cdot (x, y) := (\zeta^a \cdot x, \zeta^b \cdot y)$.

Remark 5.8. We can generalize this definition for $\mathbf{d} \in \mathbb{Z}_{>0}^r$ and $A \in \text{Mat}(r \times n; \mathbb{Z})$ with an action

$$(\zeta_1, \dots, \zeta_r) \cdot (x_1, \dots, x_n) := \left(\prod_{i=1}^r \zeta_i^{a_{i1}} x_1, \dots, \prod_{i=1}^r \zeta_i^{a_{in}} x_n \right).$$

As it is defined, the action may not be faithful but we can factorize through a quotient of the group $\mu_{d_1} \times \cdots \times \mu_{d_r}$.

Lemma 5.9. *The isotropy groups for $(\frac{1}{d}; a, b)$ are trivial (for the classes of (x, y) , $x \cdot y \neq 0$), $\mu_{\gcd(d, a)}$ (if $x \neq 0$, $y = 0$), $\mu_{\gcd(d, b)}$ (if $x = 0$, $y \neq 0$), and μ_d for the class of the origin. The underlying space of $(\frac{1}{d}; a, b)$ coincides with the underlying space of $(\frac{1}{d_1}; a_1, b_1)$ where $d_1 := \frac{d}{\gcd(a, d) \cdot \gcd(b, d)}$, $a_1 := \frac{a}{\gcd(a, d)}$, and $b_1 := \frac{b}{\gcd(b, d)}$.*

Remark 5.10. Note that $(\frac{1}{d}; a, b) = (\frac{1}{d}; ca, cb)$ if $\gcd(c, d) = 1$, and we can also replace a, b by integers congruent with them mod d . If $\gcd(d, a) = \gcd(d, b) = 1$, the origin defines the only orbit with non-trivial isotropy group and the boundary of a closed regular neighbourhood is homeomorphic to the lens space $L(d, b \cdot a^{-1} \bmod d)$, see Example 4.6. In these cases, we will usually identify the orbifold with the quotient singularity and as in Example 5.6 it is simpler to use orbifold coordinates. For example for $(\frac{1}{3}; 1, 1)$ the quotient variety is $\{(u, v, w, t) \in \mathbb{C}^4 \mid v^2 = uw, vw = ut, w^2 = vt\}$ and it is easier to handle with two *coordinates*.

While for 1-orbifolds, we have *few* bad orbifolds, it is not the case in higher dimensions.

Definition 5.11. Let $\omega := (p_x, p_y, p_z) \in \mathbb{Z}_{>0}^3$, $\gcd \omega = 1$. The ω -weighted projective plane $\mathbb{P}_\omega^2$ is the orbifold such that $|\mathbb{P}_\omega^2|$ is the quotient of $\mathbb{C}^3 \setminus \{0\}$ for the action of $\mathbb{C}^*$ defined by

$$t \cdot (x, y, z) := (t^{p_x} x, t^{p_y} y, t^{p_z} z)$$

where the orbifold charts are defined by the following maps. Let $[x : y : z]_\omega$ denote the class of (x, y, z) in the quotient, and let U_x (resp. U_y, U_z) be the open set of the quotient defined by the image of $x \neq 0$ (resp. $y \neq 0, z \neq 0$). The maps are:

$$\begin{aligned} \mathbb{C}_x^2 &\xrightarrow{\Psi_x} U_x & \mathbb{C}_y^2 &\xrightarrow{\Psi_y} U_y & \mathbb{C}_z^2 &\xrightarrow{\Psi_z} U_z \\ (y, z) &\longmapsto [1 : y : z]_\omega, & (x, z) &\longmapsto [x : 1 : z]_\omega, & (y, z) &\longmapsto [1 : y : z]_\omega. \end{aligned}$$

The orbifold charts defined by these maps are $(\mathbb{C}_a^2, U_a, \mu_{p_a}, \Psi_a)$ for $a \in \{x, y, z\}$. As open suborbifolds $U_x \cong \left(\frac{1}{p_x}; p_y, p_z\right)$, $U_y \cong \left(\frac{1}{p_y}; p_x, p_z\right)$, and $U_z \cong \left(\frac{1}{p_z}; p_x, p_y\right)$.

Remark 5.12. Let us denote $P_x := [1 : 0 : 0]_\omega$, $P_y := [0 : 1 : 0]_\omega$, $P_z := [0 : 0 : 1]_\omega$, $L_x := \{x = 0\}$, $L_y := \{y = 0\}$, and $L_z := \{z = 0\}$. The isotropy groups for $P \in \mathbb{P}_\omega^2$ are:

- If $P \in U_x \cap U_y \cap U_z = \mathbb{P}_\omega^2 \setminus (L_x \cup L_y \cup L_z)$, then its isotropy group is $\{1\}$.
- If $P \in L_x \setminus \{P_y, P_z\} =: L_x^\vee$, then its isotropy group is μ_{a_x} , where $a_x := \gcd(p_y, p_z)$.
- If $P \in L_y \setminus \{P_x, P_z\} =: L_y^\vee$, then its isotropy group is μ_{a_y} , where $a_y := \gcd(p_x, p_z)$.
- If $P \in L_z \setminus \{P_x, P_y\} =: L_z^\vee$, then its isotropy group is μ_{a_z} , where $a_z := \gcd(p_x, p_y)$.
- If $P = P_a$, $a \in \{x, y, z\}$, then its isotropy group is μ_{p_a} .

If $\eta := (b_x, b_y, b_z)$, where $b_x := \frac{p_x}{a_y a_z}$, $b_y := \frac{p_y}{a_x a_z}$, and $b_z := \frac{p_z}{a_x a_y}$, then the map

$$\begin{aligned} \mathbb{P}_\omega^2 &\longrightarrow \mathbb{P}_\eta^2 \\ [x : y : z]_\omega &\longmapsto [x^{a_x} : y^{a_y} : z^{a_z}]_\eta \end{aligned}$$

is a bijective orbifold map (a forget map).

Using dimension techniques or using Zariski-van Kampen one can prove the following result.

Lemma 5.13. $\pi_1^{\text{orb}}(\mathbb{P}_\omega^2) = 1$. In particular, if $\omega \neq (1, 1, 1)$, then $\mathbb{P}_\omega^2$ is a bad orbifold.

Example 5.14. The main objects for the projective plane are projective plane curves defined by homogeneous polynomials. For weighted projective planes. they are weighted projective plane curves defined by weighted homogeneous polynomials. Let $w = (p_x, p_y, p_z)$, $\gcd \omega = 1$, and let $F_{dp_x}(x, y, z), F_{dp_y}(x, y, z), F_{dp_z}(x, y, z)$ be three homogeneous polynomials of degrees dp_x, dp_y, dp_z , respectively, such that they have no common zero in $\mathbb{P}^2$. We can check the existence of an orbifold map as follows:

$$\begin{aligned} \mathbb{P}^2 &\longrightarrow \mathbb{P}_\omega^2 \\ [x : y : z] &\longmapsto [F_{dp_x}(x, y, z) : F_{dp_y}(x, y, z) : F_{dp_z}(x, y, z)]_\omega \end{aligned}$$

In a similar way we can define maps between weighted projective planes.

Let us consider now non-abelian quotients.

Example 5.15. For $n > 2$, we can consider the action generated by

$$\begin{aligned} \mathbb{C}^2 &\xrightarrow{\rho_\zeta} \mathbb{C}^2 & \mathbb{C}^2 &\xrightarrow{\sigma} \mathbb{C}^2 \\ (x, y) &\longmapsto (\zeta x, \zeta^{-1} y) & (x, y) &\longmapsto (y, x) \end{aligned}$$

where $\zeta \in \mu_n$. These diffeomorphisms generate the dihedral group D_{2n} of order $2n$ which has μ_n as the subgroup $\{\rho_\zeta \mid \zeta \in \mu_n\}$; note that is the usual dihedral action in different coordinates. Note that $\mathbb{C}^2_{\mu_n} = (\frac{1}{n}; 1, -1)$. The map

$$\begin{aligned} \mathbb{C}^2 &\longrightarrow \mathbb{C}^2 \\ (x, y) &\longmapsto (x^n + y^n, xy) \end{aligned}$$

is a model for the quotient. Then $|\mathbb{C}^2_{D_{2n}}| \equiv \mathbb{C}^2$ with coordinates (u, v) . The image of the diagonal is the curve $C_n : u^2 = 4v^n$. The isotropy stratification is given by

$$\underbrace{(\mathbb{C}^2 \setminus C_n)}_{\{1\}} \sqcup \underbrace{(C_n \setminus \{(0, 0)\})}_{\mu_2} \sqcup \underbrace{\{(0, 0)\}}_{D_{2n}}$$

The topological counterpart is the fact that the D_{2n} -Galois cover of $\mathbb{S}^3$ ramified along a $(2, n)$ -torus knot (or link) is $\mathbb{S}^3$ (its preimage is a Hopf link with n components, where the action is not free).

Let us projectivize Example 5.15, considering the map

$$(5.4) \quad \begin{aligned} \mathbb{P}^2 &\longrightarrow \mathbb{P}^2_{(n,2,1)} \\ [x : y : z] &\longmapsto [x^n + y^n : xy : z]_{(n,2,1)} = [u : v : w]_{(n,2,1)}. \end{aligned}$$

We can do better; the action in Example 5.15 extends naturally to an action in $\mathbb{P}^2$; we consider the orbifold $\mathbb{P}^2_{D_{2n}}$. It is not hard to see that to see that $|\mathbb{P}^2_{D_{2n}}| = |\mathbb{P}^2_{(n,2,1)}|$, and if n is even it equals also $|\mathbb{P}^2_{(\frac{n}{2},1,1)}|$. The orbifold map factorizes from the map in (5.4) via the forget map.

If n is odd, the stratification by the isotropy groups gives 5 strata: $[0 : 0 : 1]_{(n,2,1)}$ (D_{2n}), $[0 : 1 : 0]_{(n,2,1)}$ (μ_2), $[1 : 0 : 0]_{(n,2,1)}$ (μ_n), $\overline{C_n} \setminus \{[0 : 0 : 1]_{(n,2,1)}\}$ (μ_2), and the open stratum with trivial isotropy.

If n is even, the stratification by the isotropy groups gives 8 strata:

- $[0 : 0 : 1]_{(n,2,1)}$ (D_{2n}),
- $[1 : 0 : 0]_{(n,2,1)}$ (μ_n),
- $[1 : \pm 1 : 0]_{(n,2,1)}$ ($\mu_2 \times \mu_2$), two strata,
- $\{w = 0\} \setminus \{[1 : 1 : 0]_{(n,2,1)}, [1 : -1 : 0]_{(n,2,1)}, [1 : 0 : 0]_{(n,2,1)}\}$ (μ_2),
- $\{u = \pm 2v^{\frac{n}{2}}\} \setminus \{[1 : \pm 1 : 0]_{(n,2,1)}\}$ (μ_2), two strata,
- and the open stratum with trivial isotropy.

In the odd case, the inclusion of the closed orbifold defined by $w = 0$ induces an isomorphism of orbifold fundamental groups; note that $w = 0$ is isomorphic to $\mathbb{P}^1_{(2,2,n)}$.

Remark 5.16. This map allows to find all the symmetric curves in $\mathbb{P}^2$ by the action of the dihedral group.

This idea can be extended to the action of other groups. Let us consider the following action of the alternating group A_5 on $\mathbb{P}^2$. This action is constructed as follows. We start with a regular dodecahedron $\Delta_{12} \subset \mathbb{R}^3$ centered at the origin (one can safely replace the dodecahedron by the icosahedron). It is well-known that isometry group of Δ_{12} is isomorphic to A_5 and it can be obtained from a faithful representation in $\mathrm{PGL}(3; \mathbb{R}) \subset \mathrm{PGL}(3; \mathbb{C})$, which extends to an action on $\mathbb{P}^2$. An instance of this representation is

$$\begin{aligned} (1, \dots, 5) &\mapsto \begin{bmatrix} -1 & 0 & 0 \\ 0 & -c_5 & \frac{1}{2}c_5 + \frac{3}{4} \\ 0 & -1 & -c_5 \end{bmatrix} \\ (1, 2)(3, 4) &\mapsto \begin{bmatrix} -\frac{4}{5}c_5 - \frac{1}{5} & \frac{4}{5}c_5 + \frac{1}{5} & c_5 \\ \frac{6}{5}c_5 + \frac{4}{5} & -\frac{1}{5}c_5 + \frac{7}{10} & -\frac{1}{4} \\ \frac{8}{5}c_5 + \frac{2}{5} & \frac{2}{5}c_5 - \frac{2}{5} & c_5 + \frac{1}{2} \end{bmatrix} \\ (5, 3, 1) &\mapsto \begin{bmatrix} -\frac{4}{5}c_5 - \frac{1}{5} & -\frac{6}{5}c_5 + \frac{1}{5} & \frac{1}{2} \\ \frac{6}{5}c_5 + \frac{4}{5} & \frac{4}{5}c_5 + \frac{1}{5} & \frac{1}{2} \\ \frac{8}{5}c_5 + \frac{2}{5} & -\frac{8}{5}c_5 - \frac{2}{5} & 0 \end{bmatrix}, \end{aligned}$$

where $c_5 := \cos \frac{2\pi}{5}$. The fixed points and lines of the action are computed using the eigenspaces of the elements of $\rho(A_5)$. We obtain the following result:

- 6 points whose isotropy group is D_{10} .
- 10 points whose isotropy group is D_6 .
- 12 points whose isotropy group is μ_5 .
- 15 points whose isotropy group is μ_2^2 .
- 20 points whose isotropy group is μ_3 .
- 15 lines whose generic points (two of each type D_6, D_{10}, μ_2^2) have isotropy group μ_2 .

By duality, the fixed lines are in orbits of line arrangements L_6, L_{10}, L_{15} , and L_{20} (the subindex indicates the number of lines) defined by $f_m(x, y, z) = 0$. The lines in L_6 correspond to the lines joining the center of the opposite faces of Δ_{12} (these are points in $\mathbb{RP}^2$ and lines in its dual); the lines in L_{10} correspond to the lines

joining the center of the opposite vertices of Δ_{12} . The arrangement $L_{10} \cup L_6$ is called the icosididodecahedron arrangement; L_{15} is the closure of the points with isotropy group μ_2 .

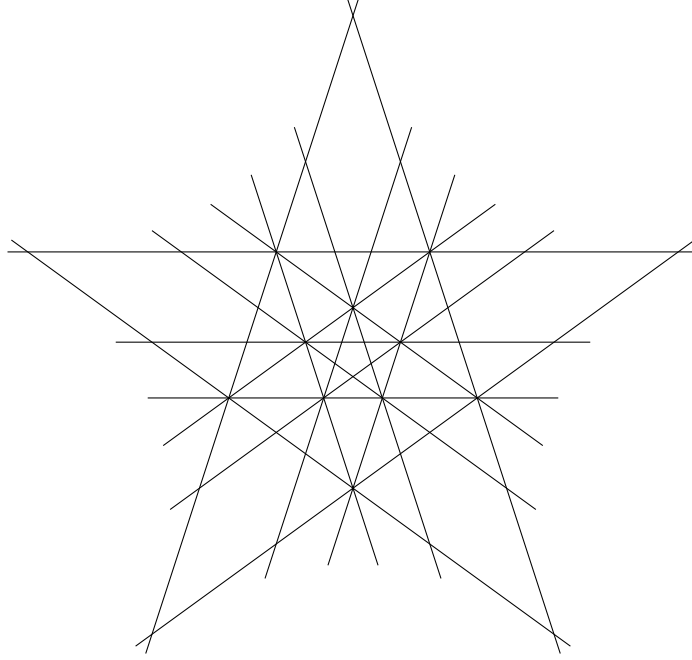


Figure 16. Affine picture of the icosidodecahedron arrangement.

In the same way, we obtain a conic invariant by the action:

$$C_2 : f_2(x, y, z) := x^2 + (1 - 2c_5)y^2 + \frac{1 - c_5}{2}z^2 = 0.$$

The map

$$\begin{aligned} \mathbb{P}^2 &\xrightarrow{\pi} \mathbb{P}_\omega^2 \\ [x : y : z] &\longmapsto [f_2 : f_6 : f_{10}]_\omega, \end{aligned}$$

where $\omega := (1, 3, 5)$ realizes the quotient. Under this map the image of the branching locus L_{15} is the curve C_{15} :

$$\begin{aligned} F_{15} := & z^3 + (65y - 4x^3)x^2z^2 - 5(144y^2 - 159x^3y + 40x^6)xyz \\ & + (1728y^3 - 3440x^3y^2 + 2275x^6y + 500x^9)y^2 = 0. \end{aligned}$$

This curve has a node at $[0 : 0 : 1]_\omega$ (smooth point). It has an $\mathbb{A}_2$ -singularity at $[1 : -\frac{5}{27} : \frac{625}{81}]$, and it has an $\mathbb{A}_4$ -singularity at $[1 : 1 : 1]_\omega$. Note that it is also tangent to $\{z = 0\}$ along $[1 : \frac{5}{8} : 0]_\omega$. The real picture in the orbifold chart $z = 1$.

One object of interest is the topology of the icosidodecahedron arrangement. An approach via this orbifold map allows to obtain meaningful data.

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