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SIGNATURES OF COLORED LINKS WITH APPLICATION TO REAL ALGEBRAIC CURVES

V. FLORENS
Universidad de Valladolid
vincent_florens@yahoo.fr

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ABSTRACT

We contruct the signature of a μ-colored oriented link, as a locally constant integer valued function with domain $(S^1 - \{1\})^\mu$. It restricts to the Tristram-Levine's signature on the diagonal and the discontinuities can occur only at the zeros of the colored Alexander polynomial. Moreover, the signature and the related nullity verify the Murasugi-Tristram inequality. This gives a new necessary condition for a link to bound a smoothly and properly embedded surface in B^4, with given Betti numbers. As an application, we achieve the classification of the complex orientations of maximal plane non-singular projective algebraic curves of degree 7, up to isotopy.

1. Introduction

The Tristram-Levine [1, 2] signature of a link, constructed in terms of a Seifert form, is a locally constant integer valued function with domain the unit circle. The discontinuities can occur only at the roots of the Alexander polynomial, whereas the related nullity is non zero only at these roots. It plays an important role in knot theory, especially to construct necessary conditions for a link to bound a smoothly and properly embedded surface with given Betti numbers in the ball B^4. Indeed, the signature is invariant by link concordance, vanishes for slice links in the strong sense [4] and the so-called Murasugi-Tristram inequality [1, 4] states that the existence of a smooth surface in B^4, with given Betti numbers and which bounds a given link, imposes restrictions on the possible values of the signature and the nullity of the link.

We define the signature and nullity for μ-colored links, as integer valued functions in a dense subset of the μ-torus. They restrict to the Tristram-Levine signature and nullity on the diagonal. For this, we mainly use Casson and Gordon constructions [7, 8] related to the twisted homology and intersection, see also [9, 10]. Roughly speaking, we apply the Atiyah-Singer α-invariant [11] to finite cyclic coverings of prime power order of a closed three-manifold related to the link complement. This extends the interpretation of Viro [12] of the Tristram-Levine signature in terms of intersection forms, and follows the spirit of [14]. We show that the signature can be extended to a locally constant function on the complement of zeros of the colored (multi-variables) Alexander polynomial, whereas the nullity is zero only on this complement. Moreover, they verify the Murasugi-Tristram inequality. This gives a new necessary condition for a link to bound a smooth non connected surface in B^4, with given Betti numbers. We also show that the signature vanishes for links with non-zero Alexander polynomial which bound a smooth surface with Euler characteristic 1. Note that, as an extension of the Fox-Milnor theorem for slice knots [5], the Alexander polynomial of such links is of the form $f\overline{f}$, for $f \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_\mu^{\pm 1}]$, see [15]. Of particular interest for these constructions is that a formula of Gilmer [17] for the Atiyah-Singer invariant allows an easy computation of the signature, in terms of a link diagram or a word in the braid group.

We apply these constructions to the study of algebraic curves. The 16^{th} problem of Hilbert requires, in particular, a classification of the oval arrangements of real non-singular algebraic curves in $\mathbb{R}P^2$, up to isotopy. It is only known for the degrees ≤ 7. Since Arnold's work, the question of the existence of such an algebraic curve with prescribed topology can be approached by studying topological properties of surfaces in $\mathbb{C}P^2$. In particular, many necessary conditions were constructed by the use of the intersections of 2-cycles in the two-fold covering of $\mathbb{C}P^2$, branched along the curve. See [13, 20] for a detailed history and references. Recently, Orevkov [21] observed that the quasipositivity of a certain braid provides a new necessary condition, and this condition is equivalent in the case of J-holomorphic curves. It implies that the link in S^3 obtained as the closure of the braid bounds a surface with given Betti numbers, smoothly and properly embedded in the ball B^4. Using the Murasugi-Tristram inequality and the Fox-Milnor theorem, Orevkov [21, 22, 24] obtained very promising results related to Hilbert's 16^{th} problem. This gives an important motivation to develop necessary conditions (or extend previous existent) in terms of link invariants.

We use the Murasugi-Tristram inequality for the signature of colored links to show that a complex orientation of an oval configuration in $\mathbb{R}P^2$ is not realizable by a maximal non-singular algebraic curve of degree 7, using Orevkov's method. This result, which achieves the classification of the complex orientations of maximal plane non-singular algebraic curves of degree 7, were not showed by previous known methods. Note that, in general, by Orevkov's method, the considered surfaces in B^4 are non-connected, at least for separating curves.

1.1. ***Historic***

The Tristram signature and nullity of an oriented link L, denoted by σ_L and η_L, are defined for all λ in the unit circle, as the signature and the nullity of the hermitian form $(1-\lambda)\theta + (1-\overline{\lambda})\theta^t$, where θ is any Seifert form of L. Note that this definition of η_L differs of 1 from Tristram's.

1.1.1. *Link concordance and Tristram-Levine signatures*

A knot is slice if it bounds a smooth 2-disk in the ball B^4, or equivalently if it is concordant to the trivial knot. Levine [2] showed that a slice knot is algebraically slice, i.e. any Seifert form of a slice knot is metabolic. It follows that the Tristram-Levine signature vanishes for slice knots. Levine also showed that the converse holds in high odd dimension, i.e. any algebraically slice knot is slice. This is false in dimension 3: Casson and Gordon [7, 8] showed that certain two-bridges knots in S^3, which are algebraically slice are not slice. For this purpose, they defined knot invariants constructed by applying the Atiyah-Singer invariant [11] to abelian coverings of the two-fold covering of the manifold obtained by a zero surgery on the knot.

Cappell and Shaneson [25] introduced a 4-dimensional viewpoint of algebraically slice knots. Roughly speaking, a knot is (4-dimensionally) algebraically slice if the infinite cyclic covering of the zero surgery manifold bounds a spin 4-manifold such that the inclusion induces an isomorphism on the first homology group and the twisted intersection form with coefficients in $\mathbb{Z}[t^{\pm 1}]$ has a metabolizer which projects onto a metabolizer for the ordinary intersection form. They show that this condition is equivalent to Levine's condition. Cochran, Teichner and Orr [26] generalized this approach and constructed a geometric filtration of the knot concordance group and an infinite sequence of new obstructions that vanish on slice knots.

In the case of links, they are several generalizations of sliceness. Fox [4] defined *slice links* in the strong sense as links concordant to the trivial link. The Tristram-Levine signature is invariant by concordance, and in particular, it vanishes for slice links. Levine [3] also constructed signatures for links such that the linking number of any two components is zero by applying the Atiyah-Singer invariant to the finite cyclic coverings of the zero surgery manifold on the link. They are concordance invariant and vanish for slice links. See also [27].

1.1.2. *The Murasugi-Tristram inequality*

The following theorem is the so-called Murasugi-Tristram inequality.

Theorem 1.1. [1, 6] *If L bounds a surface F smoothly and properly embedded in B^4 with μ connected components and rk $H_1(F) = b_1$, then, for any root of unity of prime power order λ the following inequality holds:*

$$|\sigma_L(\lambda)| + |\eta_L(\lambda) - \mu + 1| \leq b_1.$$

Murasugi [6] first proved this inequality for $\lambda = -1$ without the term $|\eta_L(\lambda) - \mu + 1|$. Tristram [1] then proved it for λ of the form $e^{2i\pi[q/2]}$ where q is a prime power. The methods of both Murasugi and Tristram are purely 3-dimensional. They relate the considered link to the trivial link by elementary transformations and study the effect of these transformations to the signatures and nullity. Viro [12] interpreted σ_L and η_L as the signatures and nullities of intersection forms related to coverings of B^4 branched along a surface with boundary L, obtained by pushing a Seifert surface from S^3. Next Kauffman and Taylor [28] gave a proof of the inequality, using coverings, in the case $\lambda = -1$. Kauffman [29] gave also a proof of it in the case of connected surfaces ($\mu = 1$), for all λ of prime power order. Gilmer [17] finally gave a complete proof using branched coverings of S^4, obtained by gluing two copies of B^4.

1.2. *Signatures of colored links*

Consider a compact closed 3-manifold M with $b_1 = b_1(M) \geq 1$ and $\tau(M)$ the Milnor torsion of M, defined as the Reidemeister torsion associated to the universal free abelian covering of M, see Section 2.

In Section 3, we construct the signature $\sigma(M)$ and the related nullity $\eta(M)$ as maps $T^{b_1}_{\mathcal{P}} \longrightarrow \mathbb{Z}$, where $T^{b_1}_{\mathcal{P}}$ is the set of *primary points* of the b_1-torus T^{b_1} (see Definition 3.14), dense in T^{b_1}. The signature is defined by considering a reformulation, due to Casson and Gordon [7, 8], of the Atiyah-Singer α-invariant [11] of finite cyclic coverings of prime power order of M.

These maps have properties closely related to $\tau(M)$. First recall that this later coincides, up to factors of the form $t_i - 1$, with the Alexander polynomial $\Delta(M)$, see [30]. We show that the signature can be extended (by continuity) to a locally constant map on the complement of $\{\tau(M) = 0\}$ in T^{b_1}, and at each primary points, the nullity is zero in this complement. See Section 3.2 and 3.3. Note that in the case $\tau(M) \neq 0$, this complement has codimension one by the symmetry of the torsion.

In Section 3.4, we show that both signatures and nullities are invariant by homology cobordism. We also show that the signature vanishes for a special class of 3-manifolds, that we call *bordant* manifolds. This class, which is invariant by homology cobordism, coincides in the case $b_1 = 1$, with 0.5-solvable (rationally) manifolds in the sense of Cochran, Teichner and Orr [26].

In Section 4, we apply these constructions to the boundary of the complement of a smoothly embedded surface in B^4 with boundary a given oriented link L and obtain isotopy invariants σ_L and η_L for links which admits a decomposition into μ sublinks with linking number zero. They are maps $T^{\mu}_{\mathcal{P}} \longrightarrow \mathbb{Z}$, which restrict to Tristram's signatures and nullities in the diagonal. Link signatures constructed in terms of finite cyclic covers of the complement of the link in S^3 were already considered in various contexts by Gilmer [17], Gordon and Litherland [33], Levine [3] and Smolinsky [34]. We show that σ_L can be extended to a locally constant map on the complement of $\{\Delta_L = 0\}$ in $T^{\mu}_* = (S^1 - \{1\})^{\mu}$, where $\Delta_L(t_1, \ldots, t_\mu)$ is the Alexander polynomial associated to the coloring of L. We note that this signature seems to be closely related to the signature that could be defined in terms of a generalized Seifert matrix (for colored links), according to a C-complex [35, 36]. For this, one may show that the generalized Seifert matrix evaluated at a primary point of T^{μ} is a matrix for the intersection form with twisted coefficients in the cyclotomic field of the complement of the C-complex pushed in B^4. This is similar to Viro's construction [12]. More generally, the generalized Seifert matrix should be closely related to the intersection form in $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_\mu^{\pm 1}]$ of this complement. Note that Cimasoni and Cooper [35, 36] have already proved that the determinant of this Seifert matrix is the Alexander polynomial of the colored link.

1.2.1. *Colored concordance*

In Section 4, the generalized signature and nullities are shown to be colored concordance invariants. The fact that the signature vanishes for slice links can be recovered as a particular case of the generalized Murasugi-Tristram inequality (see Section 1.2.2.).

We now restrict ourselves to colored links with non-zero colored Alexander polynomial. In Section 5, we define *geometrically bordant* colored links as links bounding a smooth surface in B^4 with Euler characteristic 1. In particular, geometrically bordant knots are slice knots. We show that *geometrically bordant* links are *algebraically bordant*, in the sense that some manifold derived from the link complement is bordant, see previous section. This "algebraic" class of links has vanishing signatures. The Alexander polynomial of geometrically bordant links is of the form $f \cdot \overline{f}$ for $f \in \mathbb{Z}[t_1, \ldots, t_\mu]$, see [15] and we suggest a way to show that it holds also for algebraically bordant links. In this way, algebraically bordant links are natural generalization of algebraically slice knots, in the sense of [25].

In an other hand, one may define *algebraically slice* colored links as links for which any generalized Seifert matrix, according to a C-complex [35, 36], is metabolic. We conjecture that if L is *geometrically bordant*, then it is algebraically slice and ask if, similarly to knots, this 3-dimensional point of view of sliceness is equivalent to the 4-dimensional (algebraic) point of view. Indeed, is L algebraically slice if and only if it is algebraically bordant ?

1.2.2. *Generalized Murasugi-Tristram inequality*

In Section 5.2, we show that σ_L and η_L verify the Murasugi-Tristram inequality, see Theorem 1.1 above. This gives a new obstruction to the existence of a non-connected surface $F_1 \sqcup \cdots \sqcup F_\mu$ smoothly embedded in B^4 with a given genus (or first Betti number) and boundary L. This result can be viewed as a specialization of a theorem of Gilmer ([17], Theorem 4.1): glue together two copies of B^4 along their boundary, gluing the surface F in the first copy to the cone on L on the second copy. One obtains the sphere S^4 with a configuration of colored surfaces. Applying Gilmer's Theorem 4.1 to this pair, one obtains the result. We show that the objects discussed there can be reinterpreted in ways that lead to significant simplifications of Gilmer's proof. In Section 6, we give an algorithm to compute σ_L and η_L and to check if the generalized Murasugi-Tristram inequality is satisfied for a given genus. Using this algorithm, we compute σ_L for a family of links coming from real algebraic curves (see Section 1.3). For these examples, our result is stronger than the classical Murasugi-Tristram inequality.

In conclusion, we mention recent results of Rudolph [37] and Ackbulut and Matveyev [38]. They also give necessary conditions for links to bound smooth given surfaces, that use link invariants such as the Thurston-Bennequin invariant and are derived from the adjunction inequality. It happens that they are of independent interest. For example these results can be used to show that the connected sum of two trefoils $(2,3)$ is not slice while the result of Fox and Milnor does not show it; conversely, the Murasugi-Tristram inequality or the theorem of Fox and Milnor can be used to show that the trefoil $(2,-3)$ is not slice, while the recent results do not show it.

1.3. *Real plane algebraic curves*

In this section, we state our result on algebraic curves. All the references for precise definitions and results are given in [13, 20].

A *real plane non-singular projective algebraic curve A of degree m* is an homogeneous real polynomial of degree m, without singularities, in three variables, considered up to constant factors. If P is such a polynomial, then the equation $P(x_0, x_1, x_2) = 0$, where the x_i are real, defines the set $\mathbb{R}A$ of the real points of the curve. $\mathbb{R}A$ is a closed one-dimensional manifold homeomorphic to a family of disjoint embedded S^1 in $\mathbb{R}P^2$. For a long time, according to the 16^{th} Hilbert's problem, the main question concerning the topology of real algebraic curves was how a nonsingular curve of degree m can be arranged in $\mathbb{R}P^2$, up to an isotopy. It can be formulated as the determination of which isotopy types of embeddings of family of disjoint S^1 in $\mathbb{R}P^2$ are realized by nonsingular plane real projective algebraic curves of degree m. The first prohibitions come from the topological consequences of Bezout's theorem. In particular the Harnack inequality shows that the number of components of $\mathbb{R}A$ is at most $\frac{(m-1)(m-2)}{2} + 1$.

At present this problem has been solved only for $m \leq 7$. The complete list of curves of degree 7, up to isotopy, was given by Viro. If we use his notation, they are of the form $\langle J \rangle$, $\langle J \sqcup 1\langle 1\langle 1\rangle\rangle\rangle$, and $\langle J \sqcup \beta\rangle \sqcup 1\langle\alpha\rangle\rangle$ where $\alpha+\beta \leq 14$ and $\alpha < 14$.

Following Klein, one is also interested in how the isotopy type of a curve is connected to the way the set $\mathbb{R}A$ of its real points is embedded to the set $\mathbb{C}A$ of its complex points (i.e. the set of points of the complex projective plane whose homogeneous coordinates satisfy the equation defining the curve). $\mathbb{C}A$ is an oriented smooth two-dimensional submanifold of the complex projective plane $\mathbb{C}P^2$. It is invariant under the antiholomorphic involution $conj : (z_0 : z_1 : z_2) \to (\overline{z_0} : \overline{z_1} : \overline{z_2})$. If the curve A is *maximal* (called also an *M-curve*) i.e. if the number of components of $\mathbb{R}A$ is exactly $\frac{(m-1)(m-2)}{2} + 1$, then $\mathbb{R}A$ divides $\mathbb{C}A$ into two connected pieces. The natural orientations of these two halves determine two opposite orientations on $\mathbb{R}A$ as their common boundary; these orientations of $\mathbb{R}A$ are called the *complex orientations*. The scheme of mutual arrangements of the connected components of $\mathbb{R}A$ enriched by the description of one of these complex orientations is called the *complex scheme* of the curve, following the terminology used by Viro.

The classification of complex schemes of maximal curves of degree m is solved only for $m \leq 6$. In the case $m = 7$, after the work of Orevkov and Le Touze-Fiedler [39], there remains one case, namely $\langle J \sqcup 7_+ \sqcup 3_- \sqcup 1_- \langle 2_+ \sqcup 2_- \rangle \rangle$, whose realizability was still unknown.

In Section we show

Theorem 1.2. *The complex scheme* $\langle J \sqcup 7_+ \sqcup 3_- \sqcup 1_- \langle 2_+ \sqcup 2_- \rangle \rangle$ *is not realizable by a real algebraic maximal curve of degree 7.*

Let B_n be the group of n-braids. A braid b in B_n is called *quasipositive* if $b = \prod_{i=1}^{m} a_i \sigma_1 a_i^{-1}$ for some $a_i \in B_n$. Orevkov observed that the quasipositivity of a certain braid provides a necessary condition for the realizability of a given scheme, see [21]. Applying this approach, it is proved in [22] that Theorem 1.2 would follow from the fact that none of the following 4-braids is quasipositive:

$$\sigma_1^{-3}\sigma_2^{-1}\sigma_1\sigma_2^{-\beta_1}\sigma_3^{-1}\sigma_2\sigma_3^{-\beta_2}\sigma_2^{-1}\sigma_3\sigma_2^{-\beta_3}\sigma_3^{-1}\sigma_2^2\sigma_1^2\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_3\sigma_2\sigma_1^2\sigma_2\sigma_1 \text{ in } B_4$$

$$\text{with } \beta_i \equiv 1 \mod 2, \qquad \beta_1 + \beta_2 + \beta_3 = 9.$$

These are pure braids and we denote successively the components of their closure by L_1, $L_2(1)$, $L_2(2)$, $L_2(3)$. Let $L_2 = L_2(1) \cup L_2(2) \cup L_2(3)$. The quasipositivity of these braids implies (see [21]) the existence of a planar surface (i.e. of genus 0) smoothly and properly embedded in B^4 with two connected components and boundary $L_1 \sqcup L_2$. In order to prove that L does not bound such a surface in B^4, Orevkov used the Murasugi-Tristram inequality (see Theorem 1.1 above) and excluded all the possibilities for $(\beta_1, \beta_2, \beta_3)$ except:

$$(7,1,1), (3,5,1), (1,5,3), (1,3,5), (1,1,7).$$

By computing the generalized signatures and nullities for these five links, we show the negative answer by Theorem 5.19. This proves Theorem 1.2 and achieves the classification of complex scheme realizable by real algebraic maximal non singular curves of degree 7. Note that the negative answer for three of these five links were proved with unitary representations of braid group, see [22].

2. Milnor torsion and twisted homology

This section is devoted to classical constructions and results related to the Milnor torsion of a closed 3-manifold. For more details, see [30].

Let C_∞ be a free infinite cyclic (multiplicative) group, with fixed generator t. For any positive integer r, the group ring $\mathbb{Z}[C_\infty^r]$ is identified with the Laurent polynomial ring $\Lambda = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$. Moreover, for the rest of this paper, *polynomial* will mean *Laurent polynomial*, and the equality up to a unit is denoted $\doteq$.

Let $Q\Lambda$ be the quotient field of Λ. By the inclusion, it is a Λ-module.

Since Λ is commutative, left or right Λ-modules are not distinguished and this allows the tensor product of Λ-modules to be modules as well.

2.1. *Torsion for acyclic complexes*

Let us consider the chain complex

$$C_* = (C_m \xrightarrow{\partial_{m-1}} C_{m-1} \xrightarrow{\partial_{m-2}} \cdots \xrightarrow{\partial_0} C_0),$$

where each C_i is a finite dimensional vector space over $Q\Lambda$ and ∂_i is a linear map satisfying $\partial_i \circ \partial_{i+1} = 0$ for all $i = 0, \ldots, m-2$. Suppose also that C_* is *based*, i.e. each C_i is equipped with a preferred basis, which will be denoted by c_i.

Suppose that C_* is acyclic, i.e. $H_i(C_*) = 0$ for all i. Let $B_i = Im\partial_i$. Since C_* is acyclic $B_i = Ker\partial_{i-1}$. It follows that $C_i/B_i \simeq Im\partial_{i-1} = B_{i-1}$. This is equivalent to the statement that the following sequence is exact:

$$0 \longrightarrow B_i \longrightarrow C_i \longrightarrow B_{i-1} \longrightarrow 0.$$

For each $i = 1, \ldots, m$, choose a basis b_i of B_i. Let $\widetilde{b}_{i-1} \subset C_i$ be a lift of b_{i-1} to C_i. From the above exact sequence it is clear that $b_i \cup \widetilde{b}_{i-1}$ forms a basis for C_i. Denote this basis by $b_i\widetilde{b}_{i-1}$ and the determinant of the transition matrix for the change of basis c_i to $b_i\widetilde{b}_{i-1}$ by $[b_i\widetilde{b}_{i-1}/c_i]$. One shows that the element $[b_i\widetilde{b}_{i-1}/c_i]$ is non-zero in $Q\Lambda$ and does not depend on the choice of the lift $\widetilde{b}_{i-1}$ of b_{i-1}.

Definition 2.1. *The* torsion *of the acyclic based complex C_* over Λ is defined as*

$$\tau(C_*) = \prod_{i=0}^{m} [b_i\widetilde{b}_{i-1}/c_i]^{(-1)^{i+1}} \in Q\Lambda^*.$$

The torsion $\tau(C_*)$ is independent of the choice of the bases b_i.

2.2. *Homological computation of the torsion*

Let H be a finitely generated Λ-module. A *presentation* of H is an exact sequence

$$\Lambda^m \longrightarrow \Lambda^n \longrightarrow H \longrightarrow 0,$$

where n is a positive integer and m is $0, \infty$ or a positive integer. The bases vectors in Λ^n determine a system of generators in H and each bases vector in Λ^m corresponds to a relation between these generators. A $(m \times n)$ matrix of the map $\Lambda^m \longrightarrow \Lambda^n$ with respect to the standard bases in Λ^m and Λ^n is called a *presentation matrix* of H. The *elementary ideal* of H is the ideal of Λ generated by the $(n \times n)$-minors of a presentation matrix of H. Note that it does not depend on the choice of the presentation matrix. The *order* of H is a generator of the smallest principal ideal of Λ containing the elementary ideal. It is denoted ord H.

Let us now consider a non-acyclic chain complex C_* over Λ, where each C_i is free and finitely generated Λ-module. Assume that C_* is based and that rank $H_*(C) = 0$. Since $H_*(C \otimes_\Lambda Q\Lambda) \simeq H_*(C) \otimes_\Lambda Q\Lambda$, this hypothesis is equivalent to $\dim_{Q\Lambda} H_*(C) \otimes_\Lambda Q\Lambda = 0$ or to the fact that the complex $C_* \otimes_\Lambda Q\Lambda$ is acyclic over $Q\Lambda$. Note that Λ is a Noetherian unique factorization domain and the following formula for the torsion of $C_* \otimes_\Lambda Q\Lambda$ holds.

Lemma 2.1. [31] *Theorem 4.7. If $C_* \otimes_\Lambda Q\Lambda$ is acyclic, then*

$$\tau(C_* \otimes_\Lambda Q\Lambda) = \prod_{i=0}^{m} \big(ord\ H_i(C)\big)^{(-1)^{i+1}}.$$

2.3. *Twisted chain complexes and homology*

Let X be a finite CW-complex with $b_1(X) \geq 1$ and $p : \widetilde{X} \longrightarrow X$ be its universal covering. Orient all open cells of X and the cells of $\widetilde{X}$ in such a way that the restriction of p to each cell is orientation preserving. Let $\pi = \pi_1(X, x)$ for $x \in X$. The action of π on $\widetilde{X}$ by covering transformations induces an action of π on the cellular chain groups $C_k(\widetilde{X})$. Extending this action by linearity to an action of $\mathbb{Z}[\pi]$, $C_k(\widetilde{X})$ can be considered as a $\mathbb{Z}[\pi]$-module. Note that the boundary homomorphisms are linear over $\mathbb{Z}[\pi]$. Choose a lift in $\widetilde{X}$ of each cell of X, where the cells of X are ordered in an arbitrary way, to form a basis of $C_*(\widetilde{X}; \mathbb{Z})$. It follows that $C_*(\widetilde{X}; \mathbb{Z})$ is a free based chain complex over $\mathbb{Z}[\pi]$, where the $C_i(\widetilde{X}; \mathbb{Z})$ are finitely generated.

Let $\alpha : \mathbb{Z}[\pi] \longrightarrow \Lambda$ be a ring homomorphism. The *twisted homology* of (X, α) with coefficient in Λ is the homology of the complex

$$C^\alpha(X; \Lambda) = \Lambda \otimes_\alpha C(\widetilde{X}; \mathbb{Z}).$$

In fact one may use the universal free abelian covering of X, instead of the universal covering. Let $G = H_1(X; \mathbb{Z})/TorsH_1(X; \mathbb{Z})$, where $TorsH_1(X; \mathbb{Z})$ is the subgroup of $H_1(X; \mathbb{Z})$ consisting of elements of finite order. Let $h : \mathbb{Z}[\pi] \longrightarrow \mathbb{Z}[G]$ be the projection and $\overline{p} : \overline{X} \longrightarrow X$ be the covering induced by h. As above, endow $\overline{X}$ with the CW-decomposition induced by X and the action of G on $\overline{X}$ induces an action of $\mathbb{Z}[G]$ on $C_*(\overline{X}; \mathbb{Z})$.

Consider now a ring homomorphism $\kappa : \mathbb{Z}[G] \longrightarrow \Lambda$ and set $\alpha = \kappa \circ h$. There is a Λ-isomorphism

$$C^\alpha(X; \Lambda) = \Lambda \otimes_\alpha C(\widetilde{X}; \mathbb{Z}) = \Lambda \otimes_\kappa C(\overline{X}; \mathbb{Z}) = C(\overline{X}; \mathbb{Z}) \otimes_\kappa \Lambda.$$

This equality allows us to compute the twisted homology of (X, α) directly from $\overline{X}$ and κ. The chain complex $C^\alpha(X; \Lambda)$ is now denoted $C^\kappa(X; \Lambda)$.

Definition 2.2. *The* twisted homology *of (X, κ), denoted as $H_*^\kappa(X; \Lambda)$, is the homology of the complex $C^\kappa(X; \Lambda)$.*

The twisted homology depends neither on the CW-structure on X nor on the choice of the cell orientations.

Remark 2.3. If $\kappa : G \longrightarrow C_\infty^r$ is a group homomorphism, it extends uniquely into a ring homomorphism $\mathbb{Z}[G] \longrightarrow \Lambda$ that is also denoted by κ. This convention is continued in the rest of the paper.

This construction can easily be extended to the case of pairs. If Y is a subcomplex of X, then the cellular chain complex $C_*(Y)$ is clearly a subcomplex of $C_*(X)$

and by definition $C_*(X,Y) = C_*(X)/C_*(Y)$. Obviously $\overline{p}^{-1}(Y)$ is a subcomplex of $\overline{X}$. Since the action of Λ on $\overline{X}$ preserves $\overline{p}^{-1}(Y)$, one obtains an action of Λ on $C_*(\widetilde{X}, \overline{p}^{-1}(Y))$. In this way, the cellular chain complex $C_*(\overline{X}, \overline{p}^{-1}(Y))$ is a free chain complex over Λ with a basis determined by a lift of the open oriented and ordered cells of $X - Y$ to $\overline{X} - \overline{p}^{-1}(Y)$. The twisted homology $H_*^\kappa(X, Y; \Lambda)$ is defined as the homology of the complex

$$C_*^\kappa(X, Y; \Lambda) = C_*(\overline{X}, \overline{p}^{-1}(Y); \mathbb{Z}) \otimes_\kappa \Lambda.$$

One can make similar constructions for the twisted cohomology.

Definition 2.3. *The* twisted cohomology *of (X, κ), denoted as $H^*_\kappa(X; \Lambda)$, is the homology of the complex*

$$Hom_\Lambda(C^\kappa(X; \Lambda); \Lambda).$$

2.4. *Generalities on pairs over C^r_∞*

Definition 2.4. *A pair (X, κ) over C^r_∞ is a compact smooth connected oriented manifold with $b_1(X) \geq 1$ and a surjective group homomorphism $\kappa : H_1(X) \longrightarrow C^r_\infty$.*

By a theorem of Whitehead, X has a canonical piecewise-linear structure, unique to ambient isotopy. More precisely, it is endowed with a maximal family of pl-triangulations, where any two pl-triangulations have a common linear subdivision which is pl. Endow X with the CW-decomposition induced by one of these pl-triangulations. Since X is compact, this CW-complex is finite.

Definition 2.5. *The pair (Y, κ) is the boundary of the pair $(X, \widetilde{\kappa})$ over C^r_∞ if $\partial X = Y$ and the following diagram commutes:*

$$\begin{array}{ccc} H_1(Y) & \xrightarrow{\kappa} & C^r_\infty \\ {\scriptstyle i_1}\downarrow & \nearrow_{\widetilde{\kappa}} & \\ H_1(X) & & \end{array}$$

where i_1 is the map induced by the inclusion. Note that $\widetilde{\kappa}$ is necessarily surjective. In the rest of the paper, $\widetilde{\kappa}$ is also denoted by κ.

Geometrically, that means that the covering of X induced by $\widetilde{\kappa}$ has boundary the covering of Y and the group action coincides with those given in the boundary.

Definition 2.6. *Two pairs (M_1, κ_1) and (M_2, κ_2) are* homology C^r_∞-cobordant *if there exist a pair (W, κ) such that*

. $\partial W = M_1 \sqcup -M_2$
. $H_*(W, M_i) = 0$ *for* $i = 1, 2$.
. $\kappa|_{H_1(M_i)} = \kappa_i$ *for* $i = 1, 2$.

2.5. *Milnor torsion of a pair* (M,κ)

Let (M,κ) be a pair over C^r_∞ where M is 3-dimensional. The chain complex $C^\kappa_*(M;Q\Lambda)$ over $Q\Lambda$ is based by construction, see previous section.

Definition 2.7. *If the chain complex* $C^\kappa_*(M;Q\Lambda)$ *is acyclic, then the* Milnor torsion *of* (M,κ) *is defined as*

$$\tau_\kappa(M) = \tau(C^\kappa_*(M;Q\Lambda)).$$

The torsion $\tau_\kappa(M)$ *is a non-zero element of* $Q\Lambda$*, well-defined up to sign and multiplication by monomials.*

One shows that $\tau_\kappa(M)$ is invariant under subdivision and then is independent of the choice of the triangulation. Following Section , it can be computed with homological methods. The *Alexander module* of (M,κ) is defined as the Λ-module $H^\kappa_1(M;\Lambda)$, i.e. $H_1(C_*(\widetilde{M};\mathbb{Z}))$, where $C_*(\widetilde{M};\mathbb{Z})$ is considered as a Λ-module. Since Λ is a Noetherian ring, it is finitely generated.

Definition 2.8. *The* Alexander polynomial *of* (M,κ) *is the order of the* Λ*-module* $H^\kappa_1(M;\Lambda)$ *and is denoted by* $\Delta_\kappa(M)$*.*

The Fox differential calculus [4] can be used to compute a presentation matrix of the Alexander module of (M,κ) (and thus computes $\Delta_\kappa(M)$) in terms of κ and of a presentation of the group $\pi_1(M)$.

Remark 2.4. The rational Alexander polynomial of a pair (M,κ) over C_∞ is the order of the Alexander module $H^\kappa_1(M;\mathbb{Q}[t^{\pm 1}])$. Since $\mathbb{Q}[t^{\pm 1}]$ is a principal ideal domain, the Alexander module is principal and the polynomial is the determinant of a sub-matrix of the Fox matrix.

Theorem 2.5. [40] *Suppose that* M *is a compact connected* 3*-manifold whose boundary is non-empty and consists of tori. If* $r=1$*, let* t *be a generator of* C_∞*. Then,*

$$\tau_\kappa(M) = \begin{cases} \Delta_\kappa(M)(t-1)^{-1} \text{ if } b_1(M)=1 \\ \Delta_\kappa(M) \text{ if } b_1(M)\geq 2. \end{cases}$$

Theorem 2.5 is due to Milnor but a detailed proof can also be found in [31], which uses more explicitly Lemma 2.1.

2.6. *Determinant of intersection forms over* $\mathbb{Q}[t^{\pm 1}]$

Let (M^3,κ) be a pair over C_∞. In this section we show that the (rational) Alexander polynomial of (M^3,κ) can be interpreted as the determinant of the intersection form in $H^\kappa_2(W;\mathbb{Q}[t^{\pm 1}])$, for some (W,κ) with boundary (M,κ). It follows from the fact that this intersection form provides a matrix presentation of the Alexander module.

Note that the twisted homology $\mathbb{Q}[t^{\pm 1}]$-modules $H_*^\kappa(M;\mathbb{Q}[t^{\pm 1}])$ are isomorphic to $H_*(M^\infty;\mathbb{Q})$ considered as $\mathbb{Q}[t^{\pm 1}]$-modules, where $M^\infty \longrightarrow M$ is the infinite cyclic covering induced by κ.

Definition 2.9. *The intersection form ψ^κ of a pair (W^4,κ) over C_∞ is the bilinear form, hermitian for the involution $\bar{\cdot}$*

$$\psi^\kappa : H_2^\kappa(W;\mathbb{Q}[t^{\pm 1}]) \times H_2^\kappa(W;\mathbb{Q}[t^{\pm 1}]) \longrightarrow \mathbb{Q}[t^{\pm 1}]$$
$$\text{so that } \psi^\kappa(x,y) = \sum_{i\in\mathbb{Z}} < x, t^i \cdot y > t^{-i},$$

where $<,>$ is the ordinary intersection pairing.

It is easy to prove that for each given $(x\otimes f, y\otimes g)$, the sum is finite and ψ^κ takes values in $\mathbb{Q}[t^{\pm 1}]$.

Proposition 2.1. *Any pair (M^3,κ) over C_∞ bounds a pair (W^4,κ) such that $H_1^\kappa(W;\mathbb{Q}[t^{\pm 1}]) = 0$.*

Proof. Since the bordism group $\Omega_3(B(C_\infty)) = 0$, there exist a pair (W',κ') with boundary (M,κ). In the next paragraph, we will show that one may perform a finite number of surgeries of index 2 on (W',κ') to obtain a new pair (W,κ) with the same boundary and $\pi_1(W) = 1$. A surgery of index 2 is the following process:

$$\{W' - (c \times B^3)\} \cup_{c\times S^2} \{B^2 \times S^2\}$$

for c an S^1 embedded in W'. It "kills" the corresponding homotopy class of c in W. For details on the surgery on 4-dimensional manifolds see [44].

Consider a differentiable map $f : W' \longrightarrow S^1$ which induces $\kappa : H_1(W') \longrightarrow \mathbb{Z} = H_1(S^1)$ and choose a regular value p for which $f^{-1}(p) = \Sigma$ is a connected three dimensional sub-manifold of W', with boundary S a connected surface embedded in M. Since ambient dimension is greater than three we can make surgeries on curves disjoint from Σ. We then get a 5-dimensional cobordism U with boundary $W' \coprod -W$ such that $\partial W' = \partial W = M$. Moreover, f extends to a mapping F from U where $F^{-1}(p) = \Sigma \times [0;1]$ is a regular fiber and $W - \Sigma \times \{1\}$ is simply connected. This implies that $\pi_1(W)$ is cyclic and κ' gives an isomorphism of $\pi_1(W)$ with $\mathbb{Z}$ (or C_∞). Obviously the infinite cyclic covering $W^\infty \longrightarrow W$ induced by κ is the universal covering of W. Hence $\pi_1(W^\infty) = 1$ and $H_1^\kappa(W;\mathbb{Q}[t^{\pm 1}]) = 0$. $\square$

Lemma 2.2. *Let (W,κ) with boundary (M,κ) be given by Proposition 2.1. Then $H_2^\kappa(W;\mathbb{Q}[t^{\pm 1}])$ is a free $\mathbb{Q}[t^{\pm 1}]$-module and*

$$\det \psi^\kappa \doteq \Delta_\kappa(M).$$

Proof. We show that a Gramm matrix of ψ^κ is a presentation matrix of the $\mathbb{Q}[t^{\pm 1}]$ Alexander module of (M,κ). Let W' be W with the dual cell structure and subcomplex M'. Following Milnor [40] (see also [9]), one defines sesqui-linear forms, for all $i = 0,\ldots,4$:

$$C_i^\kappa(W;\mathbb{Q}[t^{\pm 1}]) \times C_{4-i}^\kappa(W',M';\mathbb{Q}[t^{\pm 1}]) \longrightarrow \mathbb{Q}[t^{\pm 1}]$$
$$(c,c') \longmapsto \sum_{i\in\mathbb{Z}} < c, t^i \cdot c' > t^{-i},$$

They induce $\mathbb{Q}[t^{\pm 1}]$-isomorphisms

$$C_{4-i}^\kappa(W',M';\mathbb{Q}[t^{\pm 1}]) \longrightarrow \overline{\mathrm{Hom}}_{\mathbb{Q}[t^{\pm 1}]}(C_i^\kappa(W;\mathbb{Q}[t^{\pm 1}]);\mathbb{Q}[t^{\pm 1}]).$$

These isomorphisms take the differential of $C_*^\kappa(W',M';\mathbb{Q}[t^{\pm 1}])$ to the dual of the differential of $C_*^\kappa(W;\mathbb{Q}[t^{\pm 1}])$ and induce in particular a Poincare duality isomorphism

$$H_2^\kappa(W,M;\mathbb{Q}[t^{\pm 1}]) \xrightarrow{\simeq} H^2_\kappa(W;\mathbb{Q}[t^{\pm 1}]).$$

The universal coefficient theorem applied to the chain complex $C_*^\kappa(W',M';\mathbb{Q}[t^{\pm 1}])$ implies that evaluation induces an isomorphism, since $H_1^\kappa(W;\mathbb{Q}[t^{\pm 1}])=0$,

$$H^2_\kappa(W;\mathbb{Q}[t^{\pm 1}]) \simeq \overline{\mathrm{Hom}}_{\mathbb{Q}[t^{\pm 1}]}(H_2^\kappa(W;\mathbb{Q}[t^{\pm 1}]),\mathbb{Q}[t^{\pm 1}]). \qquad (*)$$

By similar arguments, since $H_1^\kappa(W,M;\mathbb{Q}[t^{\pm 1}])=0$, one shows that

$$H_2^\kappa(W;\mathbb{Q}[t^{\pm 1}]) \simeq \overline{\mathrm{Hom}}_{\mathbb{Q}[t^{\pm 1}]}(H_2^\kappa(W,M;\mathbb{Q}[t^{\pm 1}]),\mathbb{Q}[t^{\pm 1}]).$$

Since $\mathbb{Q}[t^{\pm 1}]$ is a principal ideal domain, it follows that $H_2^\kappa(W;\mathbb{Q}[t^{\pm 1}])$ is a free $\mathbb{Q}[t^{\pm 1}]$-module.

Moreover, following $(*)$, the pairing induces an unimodular pairing

$$H_2^\kappa(W;\mathbb{Q}[t^{\pm 1}]) \times H_2^\kappa(W,M;\mathbb{Q}[t^{\pm 1}]) \longrightarrow \mathbb{Q}[t^{\pm 1}].$$

To compute the determinant of $i_2 : H_2^\kappa(W;\mathbb{Q}[t^{\pm 1}]) \longrightarrow H_2^\kappa(W,M;\mathbb{Q}[t^{\pm 1}])$ induced by the inclusion, consider the following diagram with coefficients in $\mathbb{Q}[t^{\pm 1}]$:

$$\begin{array}{ccc} H_2^\kappa(W)\times H_2^\kappa(W) & \xrightarrow{\mathrm{Id}\times i_2} & H_2^\kappa(W)\times H_2^\kappa(W,M) \\ & {\scriptstyle\psi^\kappa}\searrow \quad \swarrow & \\ & \mathbb{Q}[t^{\pm 1}] & \end{array}$$

The diagonal map on the right is just the unimodular pairing considered above. It follows that

$$\det\psi^\kappa \doteq \det i_2.$$

By the exact sequence on homology with twisted coefficients in $\mathbb{Q}[t^{\pm 1}]$ of the pair (W,M), since $H_1^\kappa(W;\mathbb{Q}[t^{\pm 1}])=0$ and i_0 is an isomorphism, we get

$$H_2^\kappa(W) \xrightarrow{i_2} H_2^\kappa(W,M) \longrightarrow H_1^\kappa(M) \longrightarrow 0.$$

It follows that any matrix of i_2 is a presentation of the Alexander $\mathbb{Q}[t^{\pm 1}]$-module of (M,κ) and

$$\det i_2 \doteq \text{ ord } H_1^\kappa(M) \doteq \Delta_\kappa(M).$$

□

3. Signature and nullity invariants

3.1. *Casson-Gordon σ-invariant*

In this section, we review the construction of a Casson and Gordon [7, 8] invariant for a pair (M, ζ) over the finite cyclic group $\mathbb{C}_q$. It is a reformulation of the Atiyah-Singer α-invariant. We also recall Gilmer's construction [17] to compute it in terms of a surgery presentation of (M, ζ).

Let M be an oriented compact three manifold and $\zeta : H_1(M) \longrightarrow \mathbb{C}^*$ be a character of finite order. Let q, a positive integer, be such that the image of ζ is a cyclic subgroup of order q, generated by $\alpha = e^{2i\pi/q}$. Since $\mathrm{Hom}(H_1(M), \mathbb{C}_q) = [M, B(\mathbb{C}_q)]$, ζ induces q-fold covering $M^q \longrightarrow M$, with a choosen deck transformation denoted also by α. As ζ maps onto $\mathbb{C}_q$, the choosen deck transformation sends x to the other endpoint of the arc that begins at x and covers a loop representing an element of $(\zeta)^{-1}(\alpha)$.

The bordism group $\Omega_3(B(\mathbb{C}_q)) = \mathbb{C}_q$, and for some integer n, n disjoint copies of M bound a compact 4-manifold W over $B(\mathbb{C}_q)$. Note n can be taken to be q. Let W^q be the induced covering with the deck transformation, denoted also by α, that restricts to α on the boundary. This induces a $\mathbb{Z}[\mathbb{C}_q]$- module structure on $C_*(W^q; \mathbb{Q})$, where the multiplication by $\alpha \in \mathbb{Z}[\mathbb{C}_q]$ corresponds to the action of α on W^q. The cyclotomic field $\mathbb{Q}(\alpha)$ is a natural $\mathbb{Z}[\mathbb{C}_q]$-module.

Definition 3.10. *Let $H_*^\zeta(W; \mathbb{Q}(\alpha))$ be the* twisted homology *of (W, ζ), i.e the homology of the complex*

$$C_*(W^q; \mathbb{Z}) \otimes_{\mathbb{Z}[\mathbb{C}_q]} \mathbb{Q}(\alpha).$$

We define similarly the twisted homology of (M, ζ).

Note that we have an isomorphism

$$H_*^\zeta(W; \mathbb{Q}(\alpha)) \simeq H_*(W^q; \mathbb{Q}) \otimes_{\mathbb{Q}[\mathbb{C}_q]} \mathbb{Q}(\alpha).$$

Let $a \rightarrow \bar{a}$ denotes the involution on $\mathbb{Q}(\alpha)$ induced by complex conjugation.

Definition 3.11. *The twisted intersection form of (W, ζ) is defined as*

$$\phi^\zeta : H_2^\zeta(W; \mathbb{Q}(\alpha)) \times H_2^\zeta(W; \mathbb{Q}(\alpha)) \longrightarrow \mathbb{Q}(\alpha)$$

so that, for all a, b in $\mathbb{Q}(\alpha)$ and x, y in $H_2(W^q; \mathbb{Z})$,

$$\phi^\zeta(x \otimes a, y \otimes b) = \overline{a} b \sum_{i=1}^{q} < x, \alpha^i y > \overline{\alpha}^i.$$

Definition 3.12. *The Casson-Gordon σ-invariant of (M, ζ) and the related nullity are*

$$\sigma(M, \zeta) = \frac{1}{n}\big(\mathit{Sign}\ (\phi^\zeta) - \mathit{Sign}\ (W)\big)$$

$$\eta(M,\zeta) = \dim\ H_1^\zeta(M;\mathbb{Q}(\alpha)).$$

Let U be a closed 4-manifold, $\zeta : H_1(U) \longrightarrow \mathbb{C}_q$ and ϕ^ζ be defined as above. Modulo torsion the bordism group $\Omega_4(B(\mathbb{C}_q))$ is generated by the constant map from $CP(2)$ to $B(\mathbb{C}_q)$. If ζ is trivial, then Sign $(\phi^\zeta) =$ Sign (U). Since both signatures are invariant under cobordism, one has Sign $(\phi^\zeta) =$ Sign (U). The independence of $\sigma(M,\zeta)$ from the choice of W and n follows from this and Novikov additivity. One may see directly that these invariants do not depend on the choice of q. In this way Casson and Gordon argued that $\sigma(M,\zeta)$ is an invariant. Alternatively one may use the Atiyah-Singer G-Signature [11] theorem and Novikov additivity.

The following Lemma 3.3 will be very useful in the following section. It is a natural generalization of Theorem 3.4 [28]. Let β_i^ζ denotes the i$^{\text{th}}$ Betti number with twisted coefficient in $\mathbb{Q}(\alpha)$ and K^ζ be $H_2^\zeta(W;\mathbb{Q}(\alpha))/(\text{Radical } \phi^\zeta)$.

Lemma 3.3. *Suppose that (W,ζ) is a pair over $\mathbb{C}_q$ with b boundary components (M_i,ζ_i), where M_i is connected and ζ_i is non-zero for all i. Then the following holds*

$$dim\ K^\zeta = -\beta_3^\zeta(W) + \beta_1^\zeta(W) + \beta_2^\zeta(W) - \sum_{i=1}^{b} \eta(M_i,\zeta_i).$$

In particular, if (W,ζ) has connected boundary (M,ζ) and satisfies $\beta_3^\zeta(W) = 0$, then

$$\eta(M,\zeta) = null(\phi^\zeta) + \beta_1^\zeta(W),$$

and $\eta(M,\zeta) = 0$ implies that ϕ^ζ is non-degenerate.

The proof of Lemma 3.3 follows from the exact sequence in twisted homology of the pair $(W, \sqcup_{i=1,\dots,b} M_i)$.

We now describe a way to compute $\sigma(M,\zeta)$ in terms of a surgery presentation of (M,ζ).

Definition 3.13. *Let K be an oriented knot in S^3. Let A be an embedded annulus such that $\partial A = K \sqcup K'$ with $lk(K,K') = f$. A* p-cable on K with twist f *is defined as the union of oriented parallel copies of K lying in A such that the number of copies with the same orientation minus the number with opposite orientation is equal to p.*

Suppose that M is obtained by surgery on a link $L_1 \cup \cdots \cup L_\nu$ with framings $f_1, \dots, f_\nu$.

Remark 3.6. The linking matrix Λ with framings in the diagonal is a presentation matrix of $H_1(M)$, where the generators are the meridians of the components. For this, consider the 4-manifold W with one 0-handle and only 2-handles attached along the components of L, with respect to the framing. We have then $\partial W = M$ and $H_1(W) = 0$. It follows that the intersection form with matrix Λ is a presentation matrix of $H_1(M)$. Note that Lemma 2.2 is an analogous result, with twisted coefficients.

Let $\omega = e^{2i\pi r/q}$ with $(r,q) = 1$ and suppose that $\zeta(m_i) = \omega^{p_i}$ for a given $\vec{p} = (p_1,\ldots,p_\nu)$ in $\mathbb{Z}^\nu$ with $\gcd(p_1,\ldots,p_\nu) = 1$. The following proposition is a generalization of a formula given in [8] Lemma 3.1, where all p_i are assumed to be 1, that is given in [17] Theorem 3.6.

Proposition 3.2. *Let L' be the link with ν' components obtained from L by replacing each component by a non-empty algebraic p_i-cable with twist f_i along this component. One has*

$$\sigma(M,\zeta) = \sigma_{L'}(\omega) - \textit{Sign }(\Lambda) + 2\frac{r(q-r)}{q^2}\vec{p}^T\Lambda\vec{p},$$
$$\eta(M,\zeta) = \eta_{L'}(\omega) - \nu' + \nu.$$

3.2. *Signature of a pair* (M,κ)

Let (M^3,κ) be a pair over C^r_∞ and $\Delta_\kappa(M)$ be the Alexander polynomial of (M^3,κ). Let $\mathcal{T}^r = S^1 \times \cdots \times S^1 \subset \mathbb{C}^r$. In this section we construct the signature of (M^3,κ) and show that it can be extended to a locally constant map in the complement of $\{\Delta_\kappa(M) = 0\}$ in $\mathcal{T}^r$.

Definition 3.14. *The set $\mathcal{T}^r_{\mathcal{P}}$ of* primary points *of $\mathcal{T}^r$ is the set of elements of the form $\omega^{\vec{p}} = (\omega^{p_1},\ldots,\omega^{p_r})$ where ω is a primitive root of prime power order q, $\vec{p} = (p_1,\ldots,p_r) \in \mathbb{Z}^r$ with $\gcd(p_1,\ldots,p_r) = 1$ and $\gcd(p_i,q) = 1$. In the case $r = 1$, we simply denote $\mathcal{T}^1_{\mathcal{P}}$ by $S^1_{\mathcal{P}}$.*

Note that $\mathcal{T}^r_{\mathcal{P}}$ is dense in $\mathcal{T}^r$.

Proposition 3.3. *If $\Delta_\kappa(M)$ is not identically zero, its zero set in $\mathcal{T}^r$ is a real algebraic hypersurface of codimension 1.*

The proof follows since $\Delta_\kappa(M)$ has real coefficients and symmetry, i.e. $\overline{\Delta_\kappa(M)} = \Delta_\kappa(M)$, see [30].

For any primary point $\omega^{\vec{p}}$ in $\mathcal{T}^r_{\mathcal{P}}$, we denote the projection

$$\begin{aligned} s_{\omega^{\vec{p}}} : C^r_\infty &\longrightarrow \mathbb{C}^* \\ t_k &\longmapsto \omega^{p_k}. \end{aligned}$$

and $\kappa_{\omega^{\vec{p}}} = s_{\omega^{\vec{p}}} \circ \kappa$, for a pair (M,κ) over C^r_∞.

See Definition 3.12 for the construction of $\sigma(M,\kappa_{\omega^{\vec{p}}})$.

Theorem 3.7. *Let (M,κ) be a pair over C^r_∞. The map*

$$\begin{aligned} \sigma_\kappa(M) : \mathcal{T}^r_{\mathcal{P}} &\longrightarrow \mathbb{Z} \\ \omega^{\vec{p}} &\longmapsto \sigma(M,\kappa_{\omega^{\vec{p}}}) \end{aligned}$$

can be extended to a locally constant map on the complement of $\{\Delta_\kappa(M) = 0\}$ in $\mathcal{T}^r$.

The rest of the section is devoted to the proof of Theorem 3.7. In Paragraph , we prove it in the case of pairs over C_∞ (i.e. $r = 1$). In Paragraph , we prove it in the general case.

3.2.1. *Proof of Theorem 3.7 when $r = 1$.*

We suppose in this paragraph that $\kappa : H_1(M) \longrightarrow C_\infty$. For any $\omega \in S^1_{\mathcal{P}}$, let $\kappa_\omega = s_\omega \circ \kappa$, where s_ω sends t to ω.

For any λ in S^1, $\mathbb{Q}(\lambda)$ is a $\mathbb{Q}[t^{\pm 1}]$-module by the map $t \longmapsto \lambda$.

Definition 3.15. *For all $\lambda \in S^1$, let $H^\kappa_*(W;\mathbb{Q}(\lambda))$ be the homology of the complex*

$$C^\kappa_*(W;\mathbb{Q}(\lambda)) \overset{def}{=} C_*(\overline{W};\mathbb{Q}) \otimes_{\kappa_\lambda} \mathbb{Q}(\lambda) \simeq C_*(W^\infty) \otimes_{s_\lambda} \mathbb{Q}(\lambda),$$

where $W^\infty \longrightarrow W$ is the covering induced by κ.

Propositions 3.4 and 3.5 below were suggested by the proof of Theorem 3 in [7].

For any $\omega \in S^1_{\mathcal{P}}$ of order q, we denote the homology of (W, κ_ω) by $H^\omega_2(W;\mathbb{Q}(\alpha))$, where $\alpha = e^{2i\pi/q}$, see Definition 3.10.

Proposition 3.4. *If $H^\kappa_1(W;\mathbb{Q}[t^{\pm 1}]) = 0$, then for any ω of order q in $S^1_{\mathcal{P}}$ there is an isomorphism of $\mathbb{Q}(\alpha)$-vector spaces*

$$H^\kappa_2(W;\mathbb{Q}(\alpha)) \simeq H^\omega_2(W;\mathbb{Q}(\alpha)).$$

Proof. Let $\overline{W} \longrightarrow W$ be the universal free abelian covering of W, see Section . We have the following isomorphism of $\mathbb{Q}[t^{\pm 1}]$-modules:

$$C^\kappa_*(W;\mathbb{Q}[t^{\pm 1}]) \overset{\text{def}}{=} C_*(\overline{W};\mathbb{Q}) \otimes_\kappa \mathbb{Q}[t^{\pm 1}]) \simeq C_*(W^\infty;\mathbb{Q}).$$

Following Milnor, there is an exact sequence of $\mathbb{Q}[t^{\pm 1}]$-modules associated to the infinite cyclic covering $W^\infty \longrightarrow W^q$:

$$H^\kappa_2(W;\mathbb{Q}[t^{\pm 1}]) \overset{t^q-1}{\longrightarrow} H^\kappa_2(W;\mathbb{Q}[t^{\pm 1}]) \longrightarrow H_2(W^q;\mathbb{Q}) \longrightarrow H^\kappa_1(W;\mathbb{Q}[t^{\pm 1}]) = 0.$$

Hence, we have an isomorphism of $\mathbb{Q}[t^{\pm 1}]$-modules

$$H_2(W^q;\mathbb{Q}) \simeq \text{ coker } (t^q - 1) \simeq H^\kappa_2(W;\mathbb{Q}[t^{\pm 1}]) \otimes_{s_\omega} \mathbb{Q}[\mathbb{C}_q].$$

We obtain

$$H^\omega_2(W;\mathbb{Q}(\alpha)) \simeq H_2(W^q;\mathbb{Q}) \otimes_{\mathbb{Q}[\mathbb{C}_q]} \mathbb{Q}(\alpha) \simeq H^\kappa_2(W;\mathbb{Q}[t^{\pm 1}]) \otimes_{s_\omega} \mathbb{Q}(\alpha).$$

□

Let ψ^κ be the intersection form on $H^\kappa_2(W;\mathbb{Q}[t^{\pm 1}])$. Recall that, by the universal coefficient theorem, if $H^\kappa_1(W;\mathbb{Q}[t^{\pm 1}]) = 0$, then $H^{\kappa_\lambda}_2(W;\mathbb{Q}(\lambda)) \simeq H^\kappa_2(W;\mathbb{Q}[t^{\pm 1}]) \otimes \mathbb{Q}(\lambda)$.

Definition 3.16. *Suppose that* $H_1^\kappa(W;\mathbb{Q}[t^{\pm 1}]) = 0$. *For all* $\lambda \in S^1$, *let* ψ^λ *be the intersection form*

$$\psi^\lambda : H_2^\kappa(W;\mathbb{Q}(\lambda)) \times H_2^\kappa(W;\mathbb{Q}(\lambda)) \longrightarrow \mathbb{Q}(\lambda)$$

constructed from ψ^κ *by tensoring with* $\mathbb{Q}(\lambda)$.

Let $I(t)$ be a Gramm matrix of ψ^κ with respect to some basis in $H_2^\kappa(W;\mathbb{Q}[t^{\pm 1}])$. By Lemma 2.2, $H_2^\kappa(W;\mathbb{Q}[t^{\pm 1}])$ is a free $\mathbb{Q}[t^{\pm 1}]$-module and $\dim_{\mathbb{Q}(\lambda)} H_2^\kappa(W;\mathbb{Q}[t^{\pm 1}])\otimes \mathbb{Q}(\lambda) = \dim_{\mathbb{Q}[t^{\pm 1}]} H_2^\kappa(W;\mathbb{Q}[t^{\pm 1}])$. Hence $I(\lambda)$ is a Gramm matrix of ψ^λ with respect to the image of the chosen basis in $H_2^\kappa(W;\mathbb{Q}[t^{\pm 1}])$.

For any $\omega \in S^1_{\mathcal{P}}$, we denote the intersection form of (W,κ_ω) by ψ^ω, see Definition 3.11.

Proposition 3.5. *For any* ω *in* $S^1_{\mathcal{P}}$, *the following equality holds*

$$\psi^\omega = \phi^\omega.$$

In particular, $I(\omega)$ *is a Gramm matrix for* ϕ^ω.

Proof. Let us consider two elements of $H_2^\omega(W;\mathbb{Q}(\alpha))$. Up to the isomorphism of Proposition 3.4 they are sums of terms on the form $(x\otimes\omega^j, y\otimes\omega^k)$ where $x,y \in H_2^\kappa(W;\mathbb{Q}[t,t^{-1}])$. Let $p : W^\infty \longrightarrow W^q$ be the projection. We have

$$\begin{aligned}\psi^\omega(x\otimes\omega^j, y\otimes\omega^k) \\ = \omega^{k-j}\psi^\omega\big(x,y\big) \\ = \omega^{k-j}\sum_{i\in\mathbb{Z}} < x, t^i y > \omega^{-i}\end{aligned}$$

and we get the result by:

$$\begin{aligned}\sum_{i\in\mathbb{Z}} < x, t^i y > \omega^{-i} &= \sum_{i=0\ldots q-1} < p(x), p(t^i y) > \omega^{-i} \\ &= \sum_{i=0\ldots q-1} < p(x), \omega^i p(y) > \omega^{-i}.\end{aligned}$$

The general case is easy to deduce from this. □

We now give the proof of Theorem 3.7 in the case $r = 1$.
Let (W,κ) with boundary (M,κ) and $H_1^\kappa(W;\mathbb{Q}[t^{\pm 1}]) = 0$ be given by Proposition 2.1. The map $\sigma_\kappa(M)$ is defined as

$$\begin{aligned}S^1_{\mathcal{P}} &\longrightarrow \mathbb{Z} \\ \omega &\longmapsto \text{Sign }(\phi^\omega) - \text{Sign }(W),\end{aligned}$$

where ϕ^ω is the intersection form of (W,κ_ω), see Definition 3.12. By Proposition 3.5, for all $\omega \in S^1_{\mathcal{P}}$, one has $\psi^\omega = \phi^\omega$, where ψ^ω is the intersection form of (W,κ) evaluated at ω, see Definition 3.16. Hence, $\lambda \longmapsto \text{Sign }(\psi^\lambda) - \text{Sign }(W)$ extends $\sigma_\kappa(M)$ to S^1. Moreover, by Lemma 2.2, $\det(\psi^\kappa) \doteq \Delta_\kappa(M)$. It follows that $\det(I(\lambda)) = \Delta_\kappa(M)(\lambda)$ for all $\lambda \in S^1$, where $I(\lambda)$ is a Gramm matrix for ψ^λ. Since the entries of $I(\lambda)$ are continuous (as polynomial maps) in λ, for $\lambda \neq 1$, the result follows.

3.2.2. *Proof of Theorem 3.7 when $r > 1$.*

Let $\vec{p} = (p_1, \dots, p_r)$ be in $\mathbb{Z}^r$ with $\gcd(p_1, \dots, p_r) = 1$. Consider the map

$$s_{\vec{p}} : C_\infty^r \longrightarrow C_\infty$$
$$t_k \longrightarrow t^{p_k},$$

and define $\kappa_{\vec{p}} = s_{\vec{p}} \circ \kappa$. Since the p_i are co-prime, $\kappa_{\vec{p}}$ is surjective.

Let Ω be a connected component of the complement of $\{\Delta_\kappa(M) = 0\}$ in T^r and $\Omega_{\mathcal{P}} = \mathcal{T}^r_{\mathcal{P}} \cap \Omega$ be the primary points of Ω. Consider the curve (embedded circle) $S_{\vec{p}}$ in $\mathcal{T}^r$, defined as

$$\mathcal{S}_{\vec{p}} = \{(\lambda^{p_1}, \dots, \lambda^{p_r}); \lambda \in S^1\} \subset \mathcal{T}^r.$$

Following Theorem 2.5 above and [31] Theorem 13.3 and Remark 13.6, the Alexander polynomial of $(M, \kappa_{\vec{p}})$ is given by $\Delta_\kappa(M)(t^{p_1}, \dots, t^{p_\mu})$. It follows that there is a bijective correspondence between $\{\Delta_{\kappa_{\vec{p}}}(M) = 0\}$ and $\{\Delta_\kappa(M) = 0\} \cap S_{\vec{p}}$. Moreover, the restriction $\sigma_\kappa(M)|_{\mathcal{S}_{\vec{p}}}$ coincides with $\sigma_{\kappa_{\vec{p}}}(M)$ at each corresponding pair of primary points. By Theorem 3.7 in the case $r = 1$, it follows that for any connected component C of $S_{\vec{p}} \cap \Omega$, the restriction $\sigma_\kappa(M)|_{C \cap \Omega_{\mathcal{P}}}$ is constant. The conclusion follows from the following purely topological lemma.

Lemma 3.4. *Let Ω be an open connected subset of $\mathcal{T}^r$ and $\Omega_{\mathcal{P}} = \mathcal{T}^r_{\mathcal{P}} \cap \Omega$ be the primary points of Ω. Let σ be a map $\Omega_{\mathcal{P}} \longrightarrow \mathbb{Z}$.*

If for any $\vec{p} \in \mathbb{Z}^r$ with $\gcd(p_1, \dots, p_r) = 1$ and for any connected component C of $S_{\vec{p}} \cap \Omega$, the restriction $\sigma_{|C \cap \Omega_{\mathcal{P}}}$ is constant, then σ is constant on $\Omega_{\mathcal{P}}$.

Proof. The proof is done the case $r = 2$, since it can be easily generalized.

Let us consider the torus T^2_* as $\mathbb{R}^2/\mathbb{Z}^2$. The curves $S_{\vec{p}}$ are the projections of *primary lines* in $\mathbb{R}^2$. Note that each primary line contains a point with integer coordinates.

Consider now the problem in $\mathbb{R}^2$ and Ω as an open connected set of $\mathbb{R}^2$. Let x, y be two primary points in Ω. Since Ω is connected there exists a path $c : [0; 1] \longrightarrow \Omega$ such that $c(0) = x$ and $c(1) = y$. Let $\{B_i\}_i$ be a covering by balls of $c([0; 1])$ in Ω (note that a ball is convex). By compactness, there exist a covering of $c([0; 1])$ by a finite subset $\{B_1, \dots, B_n\}$ of $\{B_i\}$ Let us prove that σ is locally constant on the set of primary points of each ball of Ω. This will implies that $\sigma(x) = \sigma(y)$ and σ is constant on $\Omega_{\mathcal{P}}$.

Let p be a point of Ω and $r \in \mathbb{R}^*_+$ such that $B(p, r) \subset \Omega$. We prove that σ is locally constant around each primary point $a \in B(p, r)$. For this, we prove that for any primary point m 'close' to a, there exists a path joining a to m in $B(p, r)$ constituted by two segments of primary lines. Then, by hypothesis, $\sigma(m) = \sigma(a)$ and σ is constant on a neighborhood of a.

Let $D_{(0,0),a}$ be the line which passes through $(0,0)$ and a. Let $D_{(0,1),m}$ be the line which pass through $(0,1)$ and m. Let us consider the following map F :

$$F : B(p,r) \longrightarrow \mathbb{R}^2$$

$$m \longmapsto D_{(0,0),a} \bigcap D_{(0,1),m}$$

It is continuous as a rational function on the coefficients of m. Note that $F(a) = a$. The two lines $D_{(0,0),a}$ and $D_{(0,1),m}$ are primary lines. We prove that for any m close to a, $F(m)$ is in $B(p,r)$. Let $s = r - \|a-p\|$. By continuity of F, there exist $\epsilon \geq 0$ such that $\|m-a\| \leq \epsilon \Longrightarrow \|F(m)-a\| \leq s$. Then, for all m in the ball $B(a,\epsilon)$, $\|F(m)-p\| \leq \|F(m)-a\| + \|a-p\| \leq s + \|a-p\| = r$. Then, $[m; F(m)] \cup [F(m); a]$ is a polygonal path from a to m in $B(p,r)$ whose segments lie on primary lines. □

3.3. ***Nullity of a pair*** (M,κ)

Let (M,κ) be a pair over C^r_∞ with Alexander polynomial $\Delta_\kappa(M)$, as in the previous section.

Definition 3.17. *The nullity of* (M,κ) *is the map defined as*

$$\eta_\kappa(M) : \mathcal{T}^r_{\mathcal{P}} \longrightarrow \mathbb{Z}$$

$$\omega^{\vec{p}} \longmapsto \eta(M, \kappa_{\omega^{\vec{p}}}).$$

Theorem 3.8. *For all* $\omega^{\vec{p}}$ *in* $\mathcal{T}^r_{\mathcal{P}}$,

$$\Delta_\kappa(M)(\omega^{\vec{p}}) = 0 \text{ if and only if } \eta_\kappa(M)(\omega^{\vec{p}}) \neq 0$$

Proof. Let $\omega^{\vec{p}}$ be a fixed point of $\mathcal{T}^r_{\mathcal{P}}$, such that ω has (prime power) order q. By Propositions 2.1 and 2.2, the pair $(M, \kappa_{\vec{p}})$ bounds a pair $(W, \kappa_{\vec{p}})$ over C_∞ such that

$$\det \psi^{\kappa_{\vec{p}}} = \Delta_\kappa(M)(t^{p_1}, \ldots, t^{p_\mu}),$$

where $\psi^{\kappa_{\vec{p}}}$ is the intersection form on $H_2^{\kappa_{\vec{p}}}(W; \mathbb{Q}[t^{\pm 1}])$. Let $\phi^{\omega^{\vec{p}}}$ be the twisted intersection form of $(W, \kappa_{\omega^{\vec{p}}})$, see Definition 3.11. By Definition 3.16 and Proposition 3.5, if $I(t)$ is a Gramm matrix for $\psi^{\kappa_{\vec{p}}}$, then $I(\omega)$ is a Gramm matrix for $\phi^{\omega^{\vec{p}}}$. In particular, we have

$$\det \phi^{\omega^{\vec{p}}} = \Delta_\kappa(M)(\omega^{\vec{p}}). \qquad (*)$$

Since $H_1(W;\mathbb{Z}) = \mathbb{Z}$ and $\gcd(p_1, \ldots, p_\mu) = 1$ (the covering is connected), by [17] Proposition 1.5, one shows

$$H_1^{\omega^{\vec{p}}}(W; \mathbb{Q}(\alpha)) = 0.$$

Lemma 3.3 implies that null $(\phi^{\omega^{\vec{p}}}) = \eta(M, \kappa_{\omega^{\vec{p}}})$. Hence,

$$\det \phi^{\omega^{\vec{p}}} = 0 \text{ if and only if } \eta(M, \kappa_{\omega^{\vec{p}}}) > 0$$

and the result follows from $(*)$. □

3.4. *Homology C^r_∞-cobordisms*

Theorem 3.9. *If two pairs (M_1,κ_1) and (M_2,κ_2) are C^r_∞-homology cobordant, then for all $\omega^{\vec{p}} \in \mathcal{T}^r_{\mathcal{P}}$,*

$$\sigma_{\kappa_1}(M_1)(\omega^{\vec{p}}) = \sigma_{\kappa_2}(M_2)(\omega^{\vec{p}}),$$

$$\textit{and } \eta_{\kappa_1}(M_1)(\omega^{\vec{p}}) = \eta_{\kappa_2}(M_2)(\omega^{\vec{p}}).$$

The first part of Theorem 3.9 follows from [45] Theorem 9.

Proof. For the reader convenience, we give a complete proof, as a natural generalization of the proof of Theorem 3.8 of [28]. Let (W,κ) with boundary components (M_i,κ_i) be the homology cobordism. Note that $\chi(W)=0$. Let $\omega^{\vec{p}}$ be in $\mathcal{T}^r_{\mathcal{P}}$ and $\phi^{\omega^{\vec{p}}}$ be the intersection form of $(W,\kappa_{\omega^{\vec{p}}})$. Since Sign $(W)=0$, we have

$$\sigma(M_1,{\kappa_1}_{\omega^{\vec{p}}}) - \sigma(M_2,{\kappa_2}_{\omega^{\vec{p}}}) = \text{Sign } \phi^{\omega^{\vec{p}}}.$$

Let $\beta_i^{\omega^{\vec{p}}} = \text{ dim } H_i^{\omega^{\vec{p}}}(W;\mathbb{Q}(\alpha))$. Since $\chi(W)=\chi^{\kappa_{\omega^{\vec{p}}}}(W)=0$, we have

$$\beta_1^{\omega^{\vec{p}}} = \beta_2^{\omega^{\vec{p}}} - \beta_3^{\omega^{\vec{p}}}.$$

By Lemma 3.3, we obtain dim $K^{\omega^{\vec{p}}} = -\eta(M_1,{\kappa_1}_{\omega^{\vec{p}}}) - \eta(M_2,{\kappa_2}_{\omega^{\vec{p}}}) + 2\beta_1^{\omega^{\vec{p}}}$. Moreover, by Proposition 1.5 of [17], we have $H_1^{\omega^{\vec{p}}}(W,M_i;\mathbb{Q}(\alpha))) = 0$ for $i=1,2$. Hence, from the exact sequence in homology with twisted coefficients in $\mathbb{Q}(\alpha)$ of the pairs (W,M_i), one obtains

$$\beta_1^{\omega^{\vec{p}}} \le \eta(M_i,\kappa_{\omega^{\vec{p}}}) \text{ for } i=1,2.$$

This gives dim $K^{\omega^{\vec{p}}}=0$ and Sign $\phi^{\omega^{\vec{p}}}=0$. Furthermore,

$$\beta_1^{\omega^{\vec{p}}} = \eta(M_1,{\kappa_1}_{\omega^{\vec{p}}}) = \eta(M_2,{\kappa_2}_{\omega^{\vec{p}}}).$$

□

3.5. *Pairs (M,κ) with vanishing signature*

3.5.1. *The case of pairs over C_∞*

A metabolizer is a totally isotropic module direct summand of half rank.

Definition 3.18. *A pair (M,κ) over C_∞ is C_∞-bordant if:*

- *there exists a pair (W,κ) where W is spin, with boundary (M,κ).*
- *the intersection form on $H_2^\kappa(W;\mathbb{Q}[t^{\pm1}])$ has a metabolizer whose image is a metabolizer for the form in $H_2(W)$.*

Definition 3.18 is a rational version of 0.5-solvability, see [26] Theorem 8.4. The property of being C_∞-bordant is invariant by homology C_∞-cobordism, see [26] remark 8.6.

Theorem 3.10. *If (M,κ) with $\tau_\kappa(M) \neq 0$ is C_∞-bordant, then the following holds:*
- *There exists $f \in \mathbb{Q}[t^{\pm 1}]$ such that $\tau_\kappa(M) \doteq f \cdot \overline{f}$.*
- *$\sigma_\kappa(M) = 0$.*

Proof. Let (W,κ) with boundary (M,κ) be given in Definition 3.18. Let (W',κ) with boundary (M,κ) be obtained by surgeries on (W,κ) such that $H_1^\kappa(W,\mathbb{Q}[t^{\pm 1}]) = 0$, see Proposition 2.1. Let ψ^κ be the intersection form on $H_2^\kappa(W';\mathbb{Q}[t^{\pm 1}])$. By hypothesis, ψ^κ is metabolic.

Following Proposition 2.2, $\det \psi^\kappa \doteq \tau_\kappa(M)$. Hence there exists $f \in \mathbb{Q}[t^{\pm 1}]$ such that $\tau_\kappa(M) \doteq f \cdot \overline{f}$. Let $I(t)$ be a Gramm matrix for ψ^κ . From Proposition 3.5, for any $\omega \in S^1_{\mathcal{P}}$, $\sigma_\kappa(M)(\omega) = \text{Sign } I(\omega) - \text{Sign } (W)$. It follows that the signature vanishes. □

3.5.2. *The case of pairs over C^r_∞*

Definition 3.19. *A pair (M,κ) over C^r_∞ is C^r_∞-bordant if for all $\vec{p}$ in $\mathbb{Z}^r$ with* $\gcd(p_1,\dots,p_r) = 1$ *and $\tau_\kappa(M)(t^{p_1},\dots,t^{p_\mu}) \neq 0$, the pair $(M,\kappa_{\vec{p}})$ is C_∞-bordant.*

Note that by the hypothesis $\gcd(p_1,\dots,p_r) = 1$, the corresponding infinite cyclic coverings are connected.

The property of being C^r_∞-bordant is invariant by homology C^r_∞-cobordism.

Theorem 3.11. *If (M,κ) with $\tau_\kappa(M) \neq 0$ is C^r_∞-bordant, then $\sigma_\kappa(M) = 0$.*

Proof. By Theorem 3.11, the restriction of $\sigma_\kappa(M)$ to all the curves $S_{\vec{p}}$ vanishes. Hence, $\sigma_M(\kappa)$ vanishes. □

Remark 3.12. If the pair (M,κ) over C^r_∞, with $\tau_\kappa(M) \neq 0$ satisfies the hypothesis of Theorem 3.11, then by Theorem 3.10, for all $\vec{p}$ in $\mathbb{Z}^r$ with $\gcd(p_1,\dots,p_r) = 1$, there exists $f \in \mathbb{Q}[t^{\pm 1}]$ such that

$$\tau_\kappa(M)(t^{p_1},\dots,t^{p_\mu}) \doteq f \cdot \overline{f}.$$

That suggests that $\tau_\kappa(M) \doteq g \cdot \overline{g}$ for some $g \in \mathbb{Q}[t_1^{\pm 1},\dots,t_r^{\pm 1}]$, but we have not proved this.

4. Link invariants

4.1. *Colored links and concordance*

A *μ-colored link* is an oriented link L with ν components in S^3 together with a surjective map assigning to each component a color in $\{1,\dots,\mu\}$.

For $k = 1,\dots,\mu$, let L_k be the sublinks of L corresponding to all the components of L with the same color k.

Definition 4.20. *A μ-colored link L is* algebraically split *if $lk(L_k, L_{k'}) = 0$ for all $k \neq k'$.*

Definition 4.21. *A μ-colored link L bounds a surface $F = F_1 \sqcup \cdots \sqcup F_\mu$ with μ connected components if $\partial F_k = L_k$ for all $k = 1, \ldots, \mu$.*

Note that if a μ-colored link L bounds a surface F smoothly embedded in B^4, then it is algebraically split, since the intersection number $F_k \cdot F_{k'}$ vanishes in B^4. Conversely, any algebraically split μ-colored link bounds a smoothly embedded surface in B^4 (with arbitrarily high genus).

Definition 4.22. *Two μ-colored links L and L' with ν components are* μ-concordant *if there exist a smooth, oriented submanifold $T = T_1 \sqcup \cdots \sqcup T_\nu$ of $S^3 \times I$, homeomorphic to a disjoint union of ν copies of $S^1 \times I$ such that for all $i = 1, \ldots, \nu$, $T_i \cap S^3 \times 0$ and $T_i \cap S^3 \times 1$ are respectively a component of L and L' with the same color.*

The μ-concordance is an equivalence relation for μ-colored links.

4.2. *Exteriors of surfaces in* B^4

Let L be a μ-colored algebraically split link with ν components. Let F with μ connected components be a smoothly and properly embedded surface in B^4 with boundary L, see previous section.

Let $W_F = B^4 - T(F)$ be the exterior of an open tubular neighborhood of F. By the Thom isomorphism, excision, and the long exact sequence of the pair (B^4, W_F), the homology of W_F depends only on those of F; in particular, $H_1(W_F) = \mathbb{Z}^\mu$ is generated by the meridians m_k for $k = 1, \ldots, \mu$ of the connected components of F. Let us consider the pair (W_F, κ) where κ is the isomorphism:

$$\begin{aligned} \kappa : H_1(W_F) &\longrightarrow C_\infty^\mu \\ m_k &\longmapsto t_k. \end{aligned}$$

Let (M_F, κ) be the boundary of (W_F, κ). If there are no ambiguity about the surface F, we simply denote the pairs as (W, κ) and (M, κ).

Remark 4.13. Let $E = S^3 - T(L)$ be the exterior of an open tubular neighborhood of L in S^3. Consider a Seifert surface for each sublink of L, and let γ_i for $i = 1, \ldots, \nu$ be the curves where it intersects the boundary of E. The manifold M_F can be described as the gluing of E with $F \times S^1$ along their boundary, where $F \times 1$ is glued along the curves γ_i. In particular, M_F does not depend on the embbeding of F in B^4. By a similar construction to [16] Proposition 3.13, one may use a surgery presentation of M_F to describe (M_F, κ) without using W_F.

4.3. *Alexander polynomial of links*

In this section, we recall the definition of the Alexander polynomial of a μ-colored algebraically split link (with ν components). We show that it coincides with the torsion of the boundary of the complement of a surface in B^4, see previous section.

The group $H_1(E)$ is free abelian of rank ν, generated by the meridians of the components of L. Let us consider the homomorphism $\kappa : H_1(E) \longrightarrow C_\infty^\mu$ which sends the meridians of the components of the same sublink L_k to the generator t_k in C_∞^μ.

Definition 4.23. *The* Alexander polynomial *of the μ-colored link L is the element of Λ:*

$$\Delta_L(t_1, \dots, t_\mu) = \Delta_\kappa(E).$$

Up to the multiplication by a unit in Λ, it is invariant under an isotopy of L.

By Theorem 2.5, Δ_L is also the torsion of (E, κ), up to a factor $t-1$ in the case of a knot.

Proposition 4.6. *The torsion of the pair (M_F, κ) is given by the formula*

$$\tau_\kappa(M) = \begin{cases} (t-1)^{-2} \cdot \Delta_L(t) \text{ if } \nu = \mu = 1 \\ \prod_{k=1}^{\mu} (t_k - 1)^{-\chi(F_k)} \cdot \Delta_L(t_1, \dots, t_\mu). \end{cases}$$

Proof. By identifying properly $F \times S^1$ in M_F, κ induces $\rho_k : H_1(F_k \times S^1) \longrightarrow C_\infty^\mu$ and $\epsilon : H_1(E) \longrightarrow C_\infty^\mu$. We first compute separately the torsions of $(F_k \times S^1, \rho_k)$ and (E, ϵ).

Obviously ϵ sends all the meridians of the same sublink L_k to t_k. By Theorem 2.5, the torsion of (E, ϵ) is related to the Alexander polynomial as follows:

If $\mu = 1$, we set $t = t_1$.

$$\tau_\epsilon(E) = \begin{cases} \Delta_L (t-1)^{-1} \text{ if } \nu = \mu = 1 \\ \Delta_L \text{ if } \nu \geq 2. \end{cases}$$

We now compute $\tau_{\rho_k}(F_k \times S^1)$. Note that $H_1(F_k \times S^1) = H_1(F_k) \oplus < m_k >$ and ρ_k sends m_k to t_k. The Alexander polynomial $\Delta_{\rho_k}(F_k \times S^1)$ can be computed by using the Fox differential calculus. If F_k is a disk, then $\pi_1(F_k \times S^1) = C_\infty$ and $\Delta_{\rho_k}(F_k \times S^1) = 1$. If F_k is not a disk, then the group $\pi_1(F_k \times S^1)$ can be presented by $b_1(F_k) + 1$ generators $a_1, \dots, a_{b_1(F_k)}, \widetilde{m_k}$ with the relations $a_i \widetilde{m_k} = \widetilde{m_k} a_i$ for all $i = 1, \dots, b_1(F_k)$. The corresponding presentation matrix of the Alexander module $H_1^{\rho_k}(F_k \times S^1; \Lambda)$ has $b_1(F_k)+1$ columns and $b_1(F_k)$ of them have only one non-zero entry $t_k - 1$. Thus,

$$\Delta_{\rho_k}(F_k \times S^1) = (t_k - 1)^{b_1(F_k)-1} = (t_k - 1)^{-\chi(F_k)} \text{ for all } k = 1, \dots, \mu.$$

By the multiplicativity of the torsion [32], $\tau_\kappa(M)$ is the product of the torsions of the pairs (E, ϵ) and $(F_k \times S^1, \rho_k)$, and the result follows. Let $i_1 : H_1(E) \longrightarrow H_1(M_F)$ and $i_2 : H_1(F \times S^1) \simeq \oplus_{k=1}^{\mu} H_1(F_k \times S^1) \longrightarrow H_1(M_F)$ be induced by the inclusions. Thus, $\kappa \circ i_1$ is defined on $H_1(E)$ and sends all the meridians of the same sublink L_k to t_k in C_∞^μ. By Theorem 2.5, the torsion of $(E, \kappa \circ i_1)$ is $\Delta_L(t_1, \dots, t_\mu)$ if $\nu > 1$ and $(t-1)\Delta_L(t)$ if $\nu = \mu = 1$, i.e if L is a knot.

Consider the morphism

$$\kappa \circ i_2 : \oplus_{k=1}^{\mu} H_1(F_k \times S^1) = \oplus_{k=1}^{\mu} H_1(F_k) \oplus \left(\oplus_{k=1}^{\mu} < m_k >\right) \longrightarrow C_\infty^\mu$$

$$m_k \longmapsto t_k.$$

Note that $\psi \circ i_2$ is the direct sum of the characters $H_1(F_k \times S^1) \longrightarrow C_\infty$ with $m_k \longmapsto t_k$. The result follows from the multiplicativity applied to the pairs (E, ϵ) and $(F_k \times S^1, \rho_k)$. $\square$

4.4. *Link signatures and nullities*

Let (M_F, κ) be defined as in Section .

Definition 4.24. *The signature and nullity of the μ-colored algebraically split link L are defined as*

$$\sigma_L = \sigma_\kappa(M_F),$$

$$\eta_L = \eta_\kappa(M_F).$$

They are integer valued functions with domain $\mathcal{T}_{\mathcal{P}}^\mu$.

For short the twisted homology of $(., \kappa_{\omega^{\vec{p}}})$ is denoted by $H_*^{\omega^{\vec{p}}}(.; \mathbb{Q}(\alpha))$.

Proof. We show that σ_L and η_L do not depend on the surface F. By Remark 4.13, they do not depend on the embedding of F. Hence, we have to show that they do not depend on the choice of the abstract surface.

First consider the case of the signatures. Let F_0 be a planar abstract surface with μ connected components and ν boundary components. The boundary of F_0 is a collection of ν abstract colored circles, where the coloring is induced by the connected components of F_0. Suppose that F_0 is chosen in such a way that this coloring agrees with those of L. Let M_{F_0} be constructed by gluing $F_0 \times S^1$ to E, in a similar way to Remark 4.13.

The following construction is similar to those of [16, 18]. Let $\omega^{\vec{p}}$ be a fixed element of $\mathcal{T}_{\mathcal{P}}^\mu$. If we identify properly $F \times S^1$ in M_F, $\kappa_{\omega^{\vec{p}}}$ induces a character on $H_1(F \times S^1)$ which maps $H_1(F_i)$ to 1 in $\mathbb{C}_q$. Choose inductively a collection of g disjoint curves in the kernel of $\kappa_{\omega^{\vec{p}}}$ that form a metabolizer for the intersection form on $H_1(F)/H_1(\partial F)$. By taking a tubular neighborhood of these curves in F, we obtain a collection of $(S^1 \times I)$ embedded in F. Using these embeddings, attach round 2-handles $(B^2 \times I) \times S^1$ along $(S^1 \times I) \times S^1$ to the trivial cobordism $M_F \times I$ and obtain a cobordism Ω between M_F and M_{F_0}.

Let $U = W_F \cup_{M_F} \Omega$ with boundary M_{F_0}. The $\mathbb{C}_q$-covering of W_F extends uniquely to U. Note that Ω may also be viewed as the result of attaching round 1-handles to $M_{F_0} \times I$. Since the intersection form on Ω vanishes, Sign (U) = Sign $(W_F) = 0$. If M_F^q is the $\mathbb{C}_q$-covering induced by $\kappa_{\omega^{\vec{p}}}$, the $\mathbb{C}_q$-covering of Ω, restricted to each round 2-handles is $B^2 \times I \times c$ attached to the trivial cobordism $M_F^q \times I$ along $S^1 \times I \times c$. Since the $\mathbb{C}_q$- action on these handles is given by rotation on c, by a Mayer-Veitoris argument, the inclusion induces an isomorphism which preserves the intersection form:

$$H_2^{\omega^{\vec{p}}}(U;\mathbb{Q}(\alpha)) \simeq H_2^{\omega^{\vec{p}}}(W_F;\mathbb{Q}(\alpha)).$$

and the corresponding twisted signatures coincide.

Consider now the case of the nullity. Since the deck transformation of the corresponding coverings acts by identity, one has $H_1^{\omega^{\vec{p}}}(L \times S^1) = H_1^{\omega^{\vec{p}}}(F \times S^1) = 0$. Thus, by the exact sequence with twisted coefficients of the pair $(E, F \times S^1)$, there is an isomorphism

$$H_1^{\omega^{\vec{p}}}(M_F) \simeq H_1^{\omega^{\vec{p}}}(E)$$

and $\eta_L(\omega^{\vec{p}})$ depends only on L. □

Let us denote $T_*^{\mu} = \big(S^1 - \{1\}\big)^{\mu}$.

Theorem 4.14. *If $\Delta_L(t_1, \ldots, t_\mu) \neq 0$, then*
- *σ_L can be extended to a locally constant map in $T_*^{\mu} - \{\Delta_L = 0\}$.*
- *For all primary point $\omega^{\vec{p}}$ of T_*^{μ}, $\eta_L(\omega^{\vec{p}}) = 0$ if and only if $\Delta_L(\omega^{\vec{p}}) \neq 0$.*

Proof. Following Theorem 4.6, for (M, κ) defined in Section , the zeros of $\Delta_\kappa(M)$ are those of Δ_L and the curves of the form $\{1\} \times \ldots \{1\} \times S^1 \times \{1\} \times \ldots \{1\}$. In Section these curves are denoted $S_{\vec{p}}$ for $\vec{p}$ of the form $(0, \ldots, 0, p_i, 0, \ldots, 0)$. Since σ_L is the signature of (M, κ), the result follows from Theorem 3.7. The second statement is also a consequence of Theorem 3.8 and Proposition 4.6. □

Theorem 4.15. *The signature and nullity are invariant by a μ-concordance.*

Proof. Let L and L' be two algebraically split μ-colored links and T be a μ-concordance, see Definition 4.22. Let E (resp. E') be the exterior in S^3 of an open tubular neighborhood of L (resp. L'). Let $\omega^{\vec{p}}$ be a fixed element in $\mathcal{T}^{\mu}$.

Since L and L' are μ-concordant, they bound the same surfaces in B^4. Let F with boundary L and F' with boundary L' be (the images of) two embeddings of the same abstract surface in B^4. Let (M_F, κ) (resp. $(M_{F'}, \kappa)$) be the boundary of the exterior of F (resp. F') in B^4. Note that M_F (resp. $M_{F'}$) is obtained by gluing $F \times S^1$ to E (resp. $F' \times S^1$ to E').

Let W be the exterior of an open tubular neighborhood of T in $S^3 \times I$. We have

$$H_*(W, E) = H_*(W, E') = 0.$$

Moreover, κ induces characters on $H_1(E)$ and $H_1(E')$ which extend naturally to $H_1(W)$. By a Mayer-Veitoris argument, one deduces that M_F and M_F' are C_∞^{μ}-homology cobordant. Following Theorem 3.9, one has $\sigma_L = \sigma_{L'}$ and $\eta_L = \eta_{L'}$. □

5. Links and smooth surfaces in B^4

In all the section, L is a μ-colored algebraically split link.

5.1. *Bordant links*

Let L be a μ-colored link. Let F be an abstract surface with μ connected components and ν boundary components, according to the coloring of L. Following Section , we consider the pair (M_F, κ) where M_F is obtained by gluing $F \times S^1$ to the exterior of L. See in particular Remark 4.13.

Definition 5.25. *A μ-colored algebraically split link is* algebraically bordant *if (M_F, κ) is C_∞^μ-bordant.*

Lemma 5.5. *The property of being bordant is independent of the chosen surface F to construct (M_F, κ).*

Proof. Suppose that for some surface F, (M_F, κ) is C_∞^μ-bordant. We want to prove that for any other abstract surface F' with boundary L, the corresponding pair $(M_{F'}, \kappa)$ is bordant. Let us fix $\vec{p}$ in $\mathbb{Z}^r$, with $gcd(p_1, \ldots, p_\mu) = 1$ and $\kappa_{\vec{p}} : H_1(M) \to C_\infty$ be obtained by composing $t_i \mapsto t^{p_i}$ with κ. Let $(W, \kappa_{\vec{p}})$ be a pair with boundary $(M_F, \kappa_{\vec{p}})$, such that the intersection form on $H_2^{\kappa_{\vec{p}}}(W; \mathbb{Q}[t^{\pm 1}])$ has a metabolizer whose image is a metabolizer for the form on $H_2(W)$. Note that W depends on $\vec{p}$.

We mainly use constructions made in the proof of Definition 4.24, and keep the notations of this proof. Recall that F_0 is a planar abstract surface with boundary L. Consider the cobordism Ω with boundary components M_F and M_{F_0}. Let U be obtained by gluing W and Ω along M_F. We denote also by $\kappa_{\vec{p}}$ the extension of $\kappa_{\vec{p}}$ to $H_1(U)$ and its restriction to $H_1(M_{F_0})$. A Mayer-Veitoris argument shows that the inclusion induces an isomorphism which preserves the intersection form $H_2^{\kappa_{\vec{p}}}(U; \mathbb{Q}[t^{\pm 1}]) \simeq H_2^{\kappa_{\vec{p}}}(W; \mathbb{Q}[t^{\pm 1}])$. In particular, the form on $H_2^{\kappa_{\vec{p}}}(U; \mathbb{Q}[t^{\pm 1}])$ is metabolic. Using the proof of Proposition 2.1, by surgeries on $(U, \kappa_{\vec{p}})$ we obtain a cobordant pair $(U', \kappa_{\vec{p}})$ with boundary $(M_{F_0}, \kappa_{\vec{p}})$, such that the covering of U' is the universal covering. Since the form on $H_2^{\kappa_{\vec{p}}}(U'; \mathbb{Q}[t^{\pm 1}])$ has a metabolizer and it projects onto a metabolizer for the form on $H_2(U')$, the pair $(M_{F_0}, \kappa_{\vec{p}})$ is bordant. By similar arguments, it follows that $(M_{F'}, \kappa_{\vec{p}}))$ is C_∞-bordant for all $\vec{p}$. □

Remark 5.16. If L is a knot, then L is bordant (with integral coefficients) if and only if it is algebraically slice. See [25, 26].

Theorem 5.17. *Let L be a μ-colored algebraically split link with $\Delta_L \neq 0$. If L is algebraically bordant, then σ_L vanishes.*

The proof of Theorem 5.17 follows from Theorem 3.11.

Definition 5.26. *L is* geometrically bordant *if it bounds a smoothly embedded surface in B^4 with Euler characteristic* 1.

Theorem 5.18. *Let L be a μ-colored link with $\Delta_L \neq 0$. If L is geometrically bordant, then L is algebraically bordant.*

Proof. Let W_F be the exterior of an open tubular neighborhood of F in B^4 with $\chi(F) = 1$. Let

$$\kappa : H_1(W_F) \longrightarrow C_\infty^\mu$$

$$m_i \longmapsto t_i.$$

Note that $\sigma_L = \sigma(M_F, \kappa)$ where $(M_F, \kappa) = (\partial W_F, \kappa)$. We prove that (M_F, κ) is C^{μ}_{∞}-bordant, see Definition 3.19.

Since F has μ connected components, then $H_1(F) = \mathbb{Z}^{\mu-1}$. By the Thom isomorphism, excision and the long exact sequence in homology of the pair (B^4, W_F), the integral homology of W_F is:

$$H_0(W_F) = \mathbb{Z}; H_1(W_F) = \mathbb{Z}^{\mu}; H_2(W_F) = \mathbb{Z}^{\mu-1}; H_3(W_F) = H_4(W_F) = 0.$$

In particular, $\chi(W_F) = 0$.

Let $\vec{p}$ be in $\mathbb{Z}^{\mu}$ with $\gcd(p_1, \ldots, p_\mu) = 1$ and such that $\Delta_L(t^{p_1}, \ldots, t^{p_\mu}) \neq 0$. By Theorem 2.5, the complex $C_*^{\kappa_{\vec{p}}}(M; \mathbb{Q}(t))$ is acyclic, i.e $H_*^{\kappa_{\vec{p}}}(M; \mathbb{Q}(t)) = 0$. Obviously, $H_0^{\kappa_{\vec{p}}}(W; \mathbb{Q}(t)) = H_4^{\kappa_{\vec{p}}}(W; \mathbb{Q}(t)) = 0$. By the Milnor exact sequence [41] for the infinite cyclic covering $W_F^{\infty} \longrightarrow W_F$ induced by $\kappa_{\vec{p}}$, the $\mathbb{Z}[t^{\pm1}]$-module $H_3(W_F^{\infty})$ is torsion, since $H_3(W_F) = 0$. Thus, $H_3^{\kappa_{\vec{p}}}(W_F; \mathbb{Q}(t)) = 0$. Since $H_2^{\kappa_{\vec{p}}}(M; \mathbb{Q}(t)) = 0$, by the exact sequence in twisted homology of the pair (W_F, M_F), this implies $\dim H_3^{\kappa_{\vec{p}}}(W, M; \mathbb{Q}(t)) = 0$. Thus, $H_1^{\kappa_{\vec{p}}}(W; \mathbb{Q}(t)) = 0$. Since the Euler characteristic with twisted coefficients coincide with the ordinary one, one gets $H_2^{\kappa_{\vec{p}}}(W; \mathbb{Q}(t)) = 0$.

Using the proof of Proposition 2.1, make surgeries on $(W, \kappa_{\vec{p}})$ and obtain a cobordant pair $(W', \kappa_{\vec{p}})$ with boundary (M_F, κ) and such that $H_1^{\kappa_{\vec{p}}}(W; \mathbb{Q}[t^{\pm1}]) = 0$. Thus the form on $H_2^{\kappa_{\vec{p}}}(W'; \mathbb{Q}[t^{\pm1}])$ is metabolic. Moreover, since $\kappa_{\vec{p}}$ induces the universal covering of W', the metabolizer of $\psi^{\kappa_{\vec{p}}}$ maps onto a metabolizer for the ordinary form on $H_2(W)$. □

5.2. *Generalized Murasugi-Tristram inequality*

Theorem 5.19. *Suppose that the μ-colored link L bounds a surface F, smoothly and properly embedded in B^4, with μ connected components. Let b_1 the first Betti number of F. Then, for all $\omega^{\vec{p}}$ in $\mathcal{T}^{\mu}_{\mathcal{P}}$*

$$|\sigma_L(\omega^{\vec{p}})| + |\eta_L(\omega^{\vec{p}}) - \mu + 1| \leq b_1$$

Remark 5.20. Consider the knot K obtained as the connected sum of the trefoil and its mirror image. Since K is ribbon, it is slice; but the Tristram nullity $\eta_K(e^{\frac{2i\pi}{6}})$ is non zero and the inequality of Theorem 5.19 does not hold. This illustrates that the hypothesis that $\omega^{\vec{p}}$ is a primary point, in particular that ω has prime power order, is necessary in Theorem 5.19.

Theorem 5.19 provides also an obstruction for a link to be slice in the strong sense, see [4].

Corollary 5.1. *If L is a slice link, then $\sigma_L = 0$ and $\eta_L = \mu - 1$.*

The proof follows from $b_1 = 0$. This result was shown in [3] in the following sense. An elementary necessary condition for L to be slice is that the linking matrix of L is zero. If we take the maximal coloring for the link (i.e. each component has a different color), then our signatures coincides with Levine's signature.

Remark 5.21. One may show as a direct consequence of Theorem 5.19 that links with non-zero Alexander polynomial bounding smooth surfaces with Euler characteristic 1 in B^4 have vanishing re-defined signatures. For this, write the inequality at each $\omega^{\vec{p}}$ in $\mathcal{T}^{\mu}_{\mathcal{P}}$ with $\Delta_L(\omega^{\vec{p}}) \neq 0$, i.e. $\eta_L(\omega^{\vec{p}}) = 0$.

Proof. Let W_F be the exterior of a tubular neighborhood of F in B^4. By the Thom isomorphism, excision, and the long exact sequence of the pair (B^4, W_F), the integral homology of W_F depends only on those of F:

$$H_0(W_F) = \mathbb{Z} \ ; \ H_1(W_F) = \mathbb{Z}^{\mu} \ ; \ H_2(W_F) = \mathbb{Z}^{b_1} \ ; \ H_3(W_F) = H_4(W_F) = 0.$$

In particular, the Euler characteristic $\chi(W_F) = 1 - \mu + b_1$. Let $\omega^{\vec{p}}$ be in $\mathcal{T}^{\mu}$ and $\beta_i^{\omega^{\vec{p}}} = \dim H_i^{\omega^{\vec{p}}}(W; \mathbb{Q}(\alpha))$. By Lemma 3.3 one has the following estimate

$$|\sigma_L(\omega^{\vec{p}})| \leq \dim K^{\omega^{\vec{p}}} \leq \beta_1^{\omega^{\vec{p}}} - \beta_3^{\omega^{\vec{p}}} + \beta_2^{\omega^{\vec{p}}} - \eta_L(\omega^{\vec{p}}) \qquad (*)$$

Obviously, $\beta_0^{\omega^{\vec{p}}} = 0$. Since $H_3(W_F) = 0$, by [17] Proposition 1.4, $\beta_3^{\omega^{\vec{p}}} = 0$. Since the twisted homology is constructed with the cells of W_F, the Euler characteristic with twisted coefficients coincides with the ordinary one. Thus, $\chi^{\omega^{\vec{p}}} = 1 - \mu + b_1$ and

$$\beta_2^{\omega^{\vec{p}}} = \beta_1^{\omega^{\vec{p}}} + b_1 - \mu + 1.$$

Thus, $(*)$ gives

$$|\sigma_L(\omega^{\vec{p}})| + \eta_L(\omega^{\vec{p}}) \leq b_1 - \mu + 1 + 2\beta_1^{\omega^{\vec{p}}}.$$

There are two different estimation of $\beta_1^{\omega^{\vec{p}}}$.

. From the exact sequence $H_1^{\omega^{\vec{p}}}(M) \to H_1^{\omega^{\vec{p}}}(W) \to H_1^{\omega^{\vec{p}}}(W, M) \to 0$, one has

$$\beta_1^{\omega^{\vec{p}}} \leq \eta_L(\omega^{\vec{p}}).$$

One obtains

$$|\sigma_L(\omega^{\vec{p}})| - \eta_L(\omega^{\vec{p}}) + \mu - 1 \leq b_1. \qquad (1)$$

. Since $H_1(W) = \mathbb{Z}^{\mu}$, by [17], one has

$$\beta_1^{\omega^{\vec{p}}} \leq \mu - 1.$$

One obtains

$$|\sigma_L(\omega^{\vec{p}})| + \eta_L(\omega^{\vec{p}}) - \mu + 1 \leq b_1. \qquad (2)$$

The result follows from (1) and (2). □

6. Computation

The inequality of Theorem 5.19 can be used to prove that L does not bounds a smooth surface in B^4 with given Betti numbers. If there exists $\omega^{\vec{p}}$ in $T^{\mu}_{\mathcal{P}}$ such that the inequality does not hold, then the embedded surface does not exists.

Suppose that L is a μ-colored link with components, such that each sublink $L_k = L_k(1) \cup \cdots \cup L_k(\nu_\mu)$ corresponds to the components with the same color k for $k = 1, \ldots, \mu$. Note that $\sum \nu_k = \nu$.

From now on, fix a point $\omega^{\vec{p}}$ in $\mathcal{T}^\mu_{\mathcal{P}}$, such that $\omega = e^{2i\pi r/q}$ and give an explicit algorithm to compute $\sigma_L(\omega^{\vec{p}})$ and $\eta_L(\omega^{\vec{p}})$. This algorithm arises as a direct consequence of Gilmer's formula, see Theorem 3.7 [17], and Proposition 3.2. Let

$$\vec{p}_\nu = (\underbrace{p_1, \ldots, p_1}_{\nu_1}, \ldots, \underbrace{p_k, \ldots, p_k}_{\nu_k}, \ldots, \underbrace{p_\mu, \ldots, p_\mu}_{\nu_\mu}),$$

and introduce the notation :

$$\vec{f} = \big(f_1(1), \ldots, f_1(\nu_1), \ldots, f_k(1), \ldots, f_k(\nu_k), \ldots, f_\mu(1), \ldots, f_\mu(\nu_\mu)\big)$$

Theorem 6.22. *Suppose that $\vec{f}$ is a solution of the following system of simultaneous congruences and equalities, where Λ the linking matrix of L with the vector $\vec{f}$ in the diagonal:*

$$\begin{cases} \Lambda\ \vec{p}_\nu \equiv 0 \mod q & (i) \\ \sum_{i=1}^{\nu_k} f_k(i) = -\sum_{i,j;i\neq j} lk(L_k(i), L_k(j)), \text{ for all } k = 1\ldots\mu & (ii) \end{cases}$$

Let L' be a non-empty link where the component $L_k(i)$ of L is replaced by a p_k-cable on $L_k(i)$ with twist $f_k(i)$, see Definition 3.13. If ν' denotes the number of components of L', then

$$\sigma_L(\omega^{\vec{p}}) = \sigma_{L'}(\omega)$$
$$\eta_L(\omega^{\vec{p}}) = \eta_{L'}(\omega) - \nu' + \nu$$

Proof. Let W_F with boundary M_F be the exterior of an open tubular neighborhood of a surface F in B^4 with μ connected components and boundary L. Consider

$$\zeta : H_1(W_F) \longrightarrow \mathbb{C}^*$$
$$m_k \longmapsto \omega^{p_k}.$$

The character ζ induces a character on $H_1(M)_F$ and a one on $H_1(E)$ which sends all the meridians of the same sublink L_k to ω^{p_k}. The same notation ζ is used for all of them. By Definition 4.24,

$$\sigma_L(\omega^{\vec{p}}) = \sigma(M_F, \zeta) \text{ and } \eta_L(\omega^{\vec{p}}) = \eta(M_F, \zeta).$$

We now show that there exists $\vec{f}$ such that ζ induces a character on $H_1(M_L)$, where M_L is obtained by surgery on L with framing vector $\vec{f}$. Moreover, we show that $\vec{f}$ can be chosen such that (ii) holds. The formula then follows from Theorem 3.7 of [17].

Recall that if M_L is obtained by surgery on L with vector framings $\vec{f}$, then the matrix Λ with entries linking numbers of L and $\vec{f}$ in the diagonal is a presentation matrix of $H_1(M_L)$, see Remark 3.6. If $\vec{f}$ is the solution of $\Lambda\ \vec{p}_\nu \equiv 0 \mod q$, then the character denoted ζ, which sends all the meridians of the same sublink L_k to ω^{p_k}, is well-defined on $H_1(M_L)$.

The congruence $\Lambda\,\vec{p}_\nu \equiv 0 \mod q$ is a linear system in the variables $f_k(i)$ where the congruences are of the form $p_k \cdot f_k(i) \equiv a_k(i) \mod (q)$ for $a_k(i)$ in $\mathbb{Z}$. Since $(p_k, q) = 1$, p_k is invertible in $(\mathbb{Z}/q\mathbb{Z})^*$ and each equation has a solution. It follows that there exist many $\vec{f}$ which verify (i). We now show that we can choose one of them such that (ii) holds also. Let i be in $\{1, \ldots, \mu\}$. For short, we denote here by $\Gamma_{j,k}$ for $j, k = 1 \ldots \nu_i$, the linking matrix of the sublink L_i of L.

Let $f_i(1), \ldots, f_i(\nu_i - 1)$ be given as solutions of the congruences above.

Thus, by construction there exists some integers $n_1, \ldots, n_{\nu_i - 1}$ such that for all $j = 1 \ldots \nu_i - 1$, we have

$$X_j + f_i(j).p_i + \sum_{k;k\neq j} \Gamma_{j,k}.p_i = n_j.q$$

with $X_j = \sum_{k;k\neq i} lk(L_i(j), L_k).p_k$ where $lk(L_i(j), L_k)$ is the sum of the linking number of $L_i(j)$ with the components of L_k. Note that we have $\sum_{j=1}^{\nu_i} X_j = \sum_{k;k\neq i} lk(L_i, L_k).p_k = 0$. Let us consider

$$f_i(\nu_i) := -\sum_{j,k;j\neq k} \Gamma_{j,k} - f_i(1) - \cdots - f_i(\nu_i - 1)$$

We now prove that it is solution of $\Lambda.\vec{p} \equiv 0 \mod (q)$. For this, let us compute

$$X_{\nu_i} + \Big(-\sum_{j,k;j\neq k} \Gamma_{j,k} - f_i(1) - \cdots - f_i(\nu_i - 1)\Big).p_i + \sum_{k\neq \nu_i} \Gamma_{\nu_i,k}.p_i \qquad (*)$$

Note that we have:

$$-\sum_{j,k;j\neq k} \Gamma_{j,k} = -\sum_{j;j\neq 1} \Gamma_{1,j} \cdots - \sum_{j;j\neq \nu_i} \Gamma_{\nu_i,j}$$

Thus,

$$\begin{aligned}(*) &= X_{\nu_i} + \big(-\sum_{j;j\neq 1} \Gamma_{1,j} - f_i(1)\big).p_i + \cdots + \big(-\sum_{j;j\neq \nu_i - 1} \Gamma_{\nu_i - 1,j} - f_i(\nu_i - 1)\big).p_i \\ &= X_{\nu_i} + (X_1 - n_1.q) + \ldots (X_{\nu_i - 1} - n_{\nu_i - 1}.q) \\ &= -(n_1 + \cdots + n_{\nu_i - 1}).q = 0 \mod q\end{aligned}$$

7. Examples, Proof of theorem 1.2

Consider the examples of links given in Section . In all the five cases, $L = L_1 \sqcup L_2$ is a 2-colored link with $L_2 = L_2(1) \cup L_2(2) \cup L_2(3)$ and L_1 has one component. We show in this section that L cannot bound a smooth planar surface in B^4 with two connected components. Note that such a surface has a first Betti number $b_1 = 2$.

Tristram-Levine signatures

By the Murasugi-Tristram theorem (see Theorem 1.1), if L bound a planar surface, then the Levine-Tristram signatures and nullities verify the following inequality :

$$\text{For all } \omega \text{ in } S^1_{\mathcal{P}}, \qquad |\sigma_L(\omega)| + |\eta_L(\omega) - 1| \leq 2$$

The one-variable Alexander polynomial $A_L(t)$ has no roots on $S^1 \subset \mathbb{C}$. It follows that $\eta_L \equiv 0$ and σ_L is constant on S^1. The computation gives $\sigma_L \equiv \sigma_L(-1) = -1$ in the five cases. Therefore, by the inequality above, one obtains :

$$|-1| + |0 - 2 + 1| \leq 2$$

The inequality holds and one cannot conclude.

Generalized signatures

The Tristram-Levine signatures are the restriction on the diagonal in $\mathcal{T}^2$ of the generalized signatures of L. Let $\Delta_L(t,s)$ be the Alexander polynomial of L, associated to the coloring. We have to look for connected components of $T^\mu_* - \{\Delta = 0\}$ which do not intersect the diagonal, i.e. where σ_L has chances to take different values than the Tristram signatures. To compute $\Delta_L(t,s)$, we used a program written by Orevkov [23], and the program Maple as a guide to picking $\{\Delta_L = 0\}$.

Remark 7.23. The homography

$$h : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$$

$$(t,s) \longmapsto (u,v) := \left(i\frac{t+1}{t-1}, i\frac{s+1}{s-1}\right)$$

identifies T^2_* with $\mathbb{R}^2$ in $\mathbb{C}^2$. Under this identification, the conjugation has the form $(u,v) \longmapsto (-\overline{u}, -\overline{v})$. It follows that for (u,v) in $\mathbb{R}^2$, the zero set of $\Delta'_L(u,v)$ is invariant by the involution $(u,v) \longmapsto (-u,-v)$. Hence the study of $\{(u,v); u \geq 0\}$ determines the full picture by symmetry with respect to the origin.

The following table gives the values of the signatures with $\omega = e^{2i\pi r/q}$ and $\vec{p} = (p_1, p_2)$, for the five links considered.

Table 1. Values of the signature

q	29	29	29	29	29
(p_1,p_2)	(3,2)	(3,2)	(3,2)	(3,2)	(3,2)
r	6	6	11	11	5
σ	-3	-3	-3	-3	-3

By Theorem 3.4, since Δ_L is non zero at these points, $\eta_L = 0$. We give here the details of the computation of the framings in the case of the last link, see Theorem 1.2. The linking matrix of L is:

$$\Lambda = \begin{pmatrix} f_1(1) & -5 & 2 & 3 \\ -5 & f_2(1) & 2 & -1 \\ 2 & 2 & f_2(2) & 1 \\ 3 & -1 & 1 & f_2(3) \end{pmatrix}$$

The vector of framings $f = \big(f_1, f_2(1), f_2(2), f_2(3)\big)$, must verify the system of congruences and equalities:

$$\begin{cases} f_1(1) \equiv 0 \mod (49) \\ 2f_2(1) \equiv 13 \mod (49) \\ 2f_2(2) \equiv -12 \mod (49) \\ 2f_2(3) \equiv -9 \mod (49) \end{cases}$$

and

$$\begin{cases} f_1(1) = 0 \\ f_2(1) + f_2(2) + f_2(3) + 4 = 0 \end{cases}$$

The vector $f = (0, -18, -6, 20)$ is a solution of these equations. Note that the proof of Theorem 6.22 contains a method to find such solutions. Let L' be the link obtained by taking a 3-cable with framing 0 on L_1 union a 2-cable with framings respectively $-18, -6$ and 20 on each component of L_2 (see Definition 3.13). By Theorem 5.19, we obtain :

$$|-3| + |0 - 2 + 1| \leq 2$$

This inequality is obviously false and the 2-colored links do not bound a planar surface in B^4. This proves Theorem 1.2.

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