TWISTED ALEXANDER POLYNOMIALS
OF PLANE ALGEBRAIC CURVES

J.I. COGOLLUDO AND V. FLORENS

Abstract. We consider the Alexander polynomial of a plane algebraic curve twisted by a linear representation. We show that it divides the product of the polynomials of the singularity links, for unitary representations. Moreover, their quotient is given by the determinant of its Blanchfield intersection form. Specializing in the classical case, this gives a geometrical interpretation of Libgober’s divisibility Theorem. Examples show that the twisted polynomials carry more information than the classical one.

1. Introduction

Zariski [34] used the fundamental group of the complement of a plane algebraic curve to show that there exist sextics with the same combinatorics (degree of irreducible components, local type of singularities,...) but different embeddings. This comes to show that the position of singularities has an effect on the topology of the curve, and that the fundamental group is sensitive to these phenomena. Still, since there is no classification of fundamental groups of curves, and the isomorphism problem is undecidable, one cannot directly use this invariant in an effective way. The Alexander polynomial is more manageable and also sensitive to the position of singularities, see [1, 2, 13, 20, 28]. Libgober [20] showed in particular that it divides the product of the local polynomials, associated with its singular points. This result was sharpened by Degtyarev [14] which described the type of singular points that may contribute to the global polynomial. Thereafter other invariants of the Alexander modules have been considered in order to produce Zariski pairs with non isomorphic fundamental groups which cannot be distinguished by the Alexander polynomial, see for example [5, 23, 28]. Invariants related to representations of the fundamental group such as the number of epimorphisms on finite groups (also called Hall invariants) [2], or existence of dihedral coverings [5] have also been used in this context. Furthermore, the study of lower central series [4, 30] play an important role in the case of fundamental groups of line arrangements.

In knot theory, a strategy to study problems that the Alexander polynomial is not strong enough to solve is to consider non-Abelian invariants, twisted by a linear representation of the fundamental group -see, for instance [17, 24, 33]. For mutation and concordance questions, Kirk and Livingston [18, 19] recently developed their properties in the general context of CW-complexes. Their connection with lower central series, though reasonable [12], is far from being well understood.

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In this paper, we use their results to establish the relationship between the Alexander polynomial of a plane algebraic curve, twisted by a unitary representation, and the product of the local ones.

For a finite CW-complex $X$ and a finite dimensional vector space $V$ over $\mathbb{F}$, one defines the Alexander polynomials $\Delta_{\varepsilon, \rho}^1(X)$ in $\mathbb{F}[t^{\pm 1}]$, associated with a representation $\rho : \pi_1(X) \to GL(V)$ and a choice of an epimorphism $\varepsilon : \pi_1(X) \to \mathbb{Z}$. Twisted Alexander polynomials of links are obtained by applying the invariants to their exterior in $S^3$. In particular, Wada [33] considered the invariant $\Delta_{\varepsilon, \rho}(L) = \Delta_{\varepsilon, \rho}^1(L)/\Delta_{\varepsilon, \rho}^0(L)$. Given an affine algebraic curve $\mathcal{C}$, with exterior $X$ in a sufficiently large ball $\mathbb{B}^4$, one defines $\Delta_{\varepsilon, \rho}^* (\mathcal{C}) = \Delta_{\varepsilon, \rho}^1 (\mathcal{C})/\Delta_{\varepsilon, \rho}^0 (\mathcal{C})$. The related Blanchfield [9] intersection form $\varphi^{\varepsilon, \rho}(\mathcal{C})$ is defined on $H^2_{\varphi^{\varepsilon, \rho}}(X; \mathbb{F}[t^{\pm 1}])$.

In the formulas we are presenting, equalities should be understood up to units in $\mathbb{F}[t^{\pm 1}]$.

**Theorem 1.1.** Let $\mathcal{C} = C_1 \cup \cdots \cup C_r$ be an affine algebraic curve with $r$ irreducible components. Consider an epimorphism $\varepsilon : \pi_1(X) \to \mathbb{Z}$ and a unitary representation $\rho : \pi_1(X) \to GL(V)$, such that $\mathbb{F}$ is a subfield of $\mathbb{C}$, closed by conjugation.

Let $s = \# \text{Sing}(\mathcal{C})$ and consider $(S^3_k, L_k)$ the local spheres and link singularities, $k = 1, ..., s$. Let us also denote by $(\varepsilon_k, \rho_k)$ the induced representations on $\pi_1(S^3_k - L_k)$. If the Alexander modules of $L_k$ are torsion for any $k$, then

$$\alpha \cdot \prod_k \Delta_{\varepsilon_k, \rho_k}(L_k) = \Delta_{\varepsilon, \rho}^*(\mathcal{C}) \cdot \Delta_{\varepsilon, \rho}(\mathcal{C}) \cdot \det \varphi^{\varepsilon, \rho}(\mathcal{C}),$$

where $\alpha = \prod_{\ell=1}^r \det (\text{Id} - \rho(\nu_\ell))^{t(\nu_\ell)} = \chi(\mathcal{C})$, with $s_\ell = \# \text{Sing}(\mathcal{C}) \cap \mathcal{C}_\ell$, and $\nu_\ell$ a meridian of $\mathcal{C}_\ell$.

Consider now the case of $(\varepsilon, \rho)$ such that $\rho(x) = 1 \in \mathbb{Q}$ for all $x \in \pi_1(X)$, and $\varepsilon$ sends all the meridians of $\mathcal{C}$ to $1 \in \mathbb{Z}$ (or $t$ if denoted multiplicatively). We obtain the classical Alexander polynomial and Theorem 1.1 provides the following geometrical interpretation of Libgober’s Divisibility Theorem [20].

**Corollary 1.2.** Let $\Delta_{\mathcal{C}}$ be the classical Alexander polynomial of $X$, and $\Delta_{L_k}$ be the local Alexander polynomials. If $\varphi^l(\mathcal{C})$ is the intersection form with twisted coefficients in $\mathbb{Q}[t^{\pm 1}]$, then

$$(t - 1)^{2 + s - \chi(\mathcal{C})} \prod_{k=1}^r \Delta_{L_k} = \Delta_{\mathcal{C}} \cdot \Delta_{\mathcal{C}} \cdot \det \varphi^l(\mathcal{C}).$$

Going back to knot theory, Milnor [25, 26] showed that the Alexander polynomial essentially coincides with the Franz-Reidemeister torsion of the link complement. Turaev [31, 32] further developed this construction, which provided new proofs for several classical results. The first version of twisted Alexander polynomial for knots was due to Lin [24]. Wada [33] generalized it as an invariant of finitely presentable groups endowed with a representation, in terms of Fox calculus. Then Kitano [17] showed that it coincides with a torsion of the knot complement, in the acyclic case. Kirk and Livingston [18] extended his construction to the non-acyclic case, and any finite CW-complex.
On the other hand, a procedure to compute the group of a plane curve was
developed by Zariski [34] and Van Kampen [16] and expressed by Moishezon [27]
in terms of braid monodromy. Libgober [22] showed that the complement of the
(affine) curve has the homotopy type of the 2-dimensional complex corresponding
to this presentation of the group. From this we observe relations between the twisted
Alexander polynomial and the related torsion, similar to those in knot theory. We
apply the duality theorem, due to Kirk and Livingston [18] in this context, and
obtain the main result, see Theorem 5.6.

We tried to write a paper as self-contained as possible. All the basics involved
here and the duality theorem can be found in [18]. In Section 2 we recall basic
constructions on the Reidemeister torsion of a CW-complex, in the non-acyclic case,
and the relations with twisted Alexander polynomials. In Section 3 we recall the
duality theorem for twisted torsion, due to [26] in the classical case and to [18] in
the twisted case. In Section §4, we give a brief historical approach on the twisted
torsion and Alexander polynomials of links. Section 5 is devoted to plane curves,
and to the proof of both Theorem 5.6 and Corollary 5.8. Finally, Section 6 will
illustrate with some examples the use of twisted Alexander polynomials for curves.

Note finally that a several-variable version of twisted polynomials could have
been considered, for an $\varepsilon$ related to the universal Abelian covering. For technical
reasons, since $\mathbb{F}[t^\pm 1]$ is a principal ideal domain, we restrict ourselves to this case.

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2. Twisted Alexander polynomials

2.1. Torsion of a chain complex. Let $C_*$ be a finite chain complex:

$C_* = C_n \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0,$

where $C_i$ are finite dimensional $\mathbb{F}$-vector spaces, and $\partial \circ \partial = 0$. Choose a basis $c_i$
for $C_i$ and $h_i$ a basis for the homology $H_i(C_*)$. Note that if $H_i(C_*) = 0$ for all $i$,
the complex is called acyclic. Let $b_i$ be a basis of the image of $\partial : C_{i+1} \rightarrow C_i$
and $\tilde{b}_i$ be a lift of $b_i$ in $C_{i+1}$. One easily checks that $b_i h_i \tilde{b}_{i-1}$ is a basis of $C_i$. Denote
by $[u, v]$ the determinant of the transition matrix from $u$ to $v$.

Definition 2.1. The torsion of $(C_*, c, h)$ is

$$\tau(C_*, c, h) = \prod_{i=0}^n [b_i h_i \tilde{b}_{i-1} | c_i ]^{(-1)^{i+1}} \in \mathbb{F}^*/\{\pm 1\}.$$  

Note that in the literature the torsion is sometimes defined as the inverse of $\tau(C_*, c, h)$. The torsion does not depend on the choice of $b$ and their lifts. It
depends on the choice of $c$ and $h$ as follows:

$$\tau(C_*, c', h') = \tau(C_*, c, h) \prod_i \left( \frac{[h'_i | h_i]}{[c'_i | c_i]} \right)^{(-1)^{i+1}}.$$  

The following classical lemma is very useful for computations.
Lemma 2.2. Consider a short exact sequence of complexes
\[ 0 \to C' \to C \to C'' \to 0, \]
where the complexes and their homology are based, with compatible bases. Let \( H \), with torsion \( \tau(H) \), be the related long exact sequence in homology, viewed as a based acyclic complex. One has
\[ \tau(C) = \tau(C''\tau(C''). \]

2.2. Twisted chain complexes. In this section, \( X \) is a finite CW-complex, with \( \pi = \pi_1(X, x) \) for \( x \in X \). Let us fix an epimorphism
\[ \varepsilon : \pi \to \mathbb{Z}. \]
Note that \( \varepsilon \) extends naturally to an epimorphism of algebras \( \varepsilon : \mathbb{Z}[\pi] \to \mathbb{Z}[\mathbb{Z}] \), which will be also denoted by \( \varepsilon \). We identify \( \mathbb{Z}[\pi] \) with \( \mathbb{Z}[\pm 1] \). Consider now an \( \mathbb{F} \)-vector space \( V \) of finite dimension and a representation \( \rho : \pi \to \text{GL}(V) \).

If \( \tilde{X} \to X \) denotes the universal covering, the cellular chain complex \( C_\ast(\tilde{X}; \mathbb{F}) \) is a \( \mathbb{F}[\pi] \)-module generated by the lifts of the cells of \( X \). Consider the \( \mathbb{F}[\pi] \)-module \( \mathbb{F}[\pm 1] \otimes_\mathbb{F} V \), where the action is induced by \( \varepsilon \otimes \rho \), as follows:
\[ (p \otimes v) \cdot \alpha = (p\varepsilon(\alpha)) \otimes (\rho(\alpha)v), \quad \alpha \in \pi. \]

Let the chain complex of \( (X, \varepsilon, \rho) \) be defined as the complex of \( \mathbb{F}[\pm 1] \)-modules:
\[ C^\varepsilon_\ast(\mathbb{F}, \mathbb{F}[\pm 1]) = (\mathbb{F}[\pm 1] \otimes V) \otimes_{\mathbb{F}[\pi]} C_\ast(\tilde{X}; \mathbb{F}). \]

It is a free based complex, where a basis is given by the elements of the form
\[ 1 \otimes e_i \otimes c_k, \quad \text{where} \quad \{e_i\} \text{ is a basis of } V \quad \text{and} \quad \{c_k\} \text{ is a basis of the } \mathbb{F}[\pi]-\text{module } C_\ast(\tilde{X}; \mathbb{F}), \text{ obtained by lifting cells of } X. \]

A geometrical interpretation of \( C^\varepsilon_\ast(\mathbb{F}, \mathbb{F}[\pm 1]) \) was given in [18]. We briefly recall their point of view. Consider \( X^\infty \) the infinite cyclic covering induced by \( \varepsilon \). For \( \pi = \text{Ker } \varepsilon = \pi_1(X^\infty) \), the representation \( \rho \) restricts to
\[ \overline{\rho} : \pi \to \text{GL}(V). \]

The chain complex of \( \mathbb{F}[\pi] \)-modules
\[ C^\varepsilon_\ast(\infty; \mathbb{F}) = V \otimes_{\mathbb{F}[\pi]} C_\ast(\tilde{X}) \]
can be considered as a complex of \( \mathbb{F}[\pm 1] \)-modules, \( \mathbb{F}[\pm 1] \) is a trivial \( \mathbb{F}[\pi] \)-module, as follows:
\[ t^n \cdot (v \otimes c) = v\gamma^{-n} \otimes \gamma^n c \]
where \( \gamma \in \pi \) verifies \( \varepsilon(\gamma) = t \). In [18] it is shown that \( C^\varepsilon_\ast(\infty; \mathbb{F}) \) and \( C^\varepsilon_\ast(\mathbb{F}[\pm 1]) \) are isomorphic as \( \mathbb{F}[\pm 1] \)-modules, see Theorem 2.1.

The following definition fixes some vocabulary. Denote by \( \mathbb{F}(t) \) the fraction field of \( \mathbb{F}[\pm 1] \), and define \( C^\varepsilon_\ast(\mathbb{F}, \mathbb{F}(t)) \) as follows:
\[ C^\varepsilon_\ast(\mathbb{F}, \mathbb{F}(t)) = C^\varepsilon_\ast(\mathbb{F}[\pm 1]) \otimes_{\mathbb{F}[\pm 1]} \mathbb{F}(t). \]

Definition 2.3. \( (X, \varepsilon, \rho) \) is acyclic if the chain complex \( C^\varepsilon_\ast(\mathbb{F}, \mathbb{F}(t)) \) is acyclic over \( \mathbb{F}(t) \). The classical case corresponds to the case of a trivial \( \rho \), i.e. if \( V = \mathbb{F} = \mathbb{Q} \) and \( \rho(x) = 1 \) for all \( x \in \pi \).
2.3. Torsion and Alexander polynomials. Suppose that we are given \((X, \varepsilon, \rho)\),
as in previous section. Recall that the chain complex \(C^\varepsilon_\rho(X, F(t))\) is based by
construction.

**Definition 2.4.** Fix a basis for the homology \(H_*^{\varepsilon, \rho}(X; F(t))\). Let \(\tau_{\varepsilon, \rho}(X)\) be the
torsion of \((X, \varepsilon, \rho)\) with respect to this basis :

\[
\tau_{\varepsilon, \rho}(X) = \tau(C^\varepsilon_\rho(X, F(t))) \in F(t)^* ,
\]

Up to multiplication by a factor \(ut^n\) with \(u \in F^*\) and \(n \in \mathbb{Z}\), the torsion \(\tau_{\varepsilon, \rho}(X)\) is
independent on the choice of the bases, and it is a well-defined invariant of \((X, \varepsilon, \rho)\).

As it was mentioned in [18], the indeterminacy of \(\tau_{\varepsilon, \rho}(X)\) could be reduced. The
reason for not doing so can be found in Theorem 2.8 below.

**Definition 2.5.** The homology of \((X, \varepsilon, \rho)\) is defined as the \(F[t^{\pm 1}]\)-modules

\[
H_*^{\varepsilon, \rho}(X; F[t^{\pm 1}]) = H_*(C^\varepsilon_\rho(X, F[t^{\pm 1}])).
\]

One extends the definition to \(H_*^{\varepsilon, \rho}(X; F(t))\) = \(H_*(C^\varepsilon_\rho(X, F(t)))\). Since \(F[t^{\pm 1}]\) is
a principal ideal domain, and \(F(t)\) is flat over \(F[t^{\pm 1}]\), one has

\[
H_*^{\varepsilon, \rho}(X; F(t)) \simeq H_*^{\varepsilon, \rho}(X; F[t^{\pm 1}]) \otimes F(t). \quad (\ast)
\]

In particular, \(H_*^{\varepsilon, \rho}(X; F(t))\) are \(F(t)\)-vector spaces.

Since \(F[t^{\pm 1}]\) is a principal ideal domain, any \(F[t^{\pm 1}]\)-module \(H\) can be decomposed
as a direct sum of cyclic modules. The order of \(H\) is the product of the generators
of the torsion part. If the module is free the order is 1, by convention. Note that
the order of \(H\) is defined up to multiplication \(ut^n\) for \(u \in F^*\).

**Remark 2.6.** Note that, if \(\rho = \rho_1 \oplus \rho_2\) is a non-irreducible representation, then

\[
H_*^{\varepsilon, \rho}(X; F[t^{\pm 1}]) = H_*^{\varepsilon, \rho_1}(X; F[t^{\pm 1}]) \oplus H_*^{\varepsilon, \rho_2}(X; F[t^{\pm 1}]),
\]

and hence

\[
\tau_{\varepsilon, \rho}(X) = \tau_{\varepsilon, \rho_1}(X) \tau_{\varepsilon, \rho_2}(X).
\]

Therefore in the sequel we will only consider irreducible representations unless
otherwise stated.

**Definition 2.7.** The \(i\)-th Alexander polynomial \(\Delta_i^{\varepsilon, \rho}(X)\) of \((X, \varepsilon, \rho)\) is
\(H_*^{\varepsilon, \rho}(X; F[t^{\pm 1}])\).

For short, we denote \(\Delta_{\varepsilon, \rho}(X) = \Delta_{\varepsilon, \rho}(X) \frac{1}{\Delta_{\varepsilon, \rho}(X)}\) the element of \(F(t)\).

Note that in general, \(\Delta_{\varepsilon, \rho}(X)\) may not be a polynomial.

**Theorem 2.8** ([17, 18, 31]). Let \(\tau_{\varepsilon, \rho}(X)\) be the torsion of \((X, \varepsilon, \rho)\) with respect to
some basis in homology. One has

\[
\tau_{\varepsilon, \rho}(X) = \prod_i \frac{\Delta_{\varepsilon, \rho}^{2i+1}(X)}{\prod_i \Delta_{\varepsilon, \rho}^{2i}(X)}.
\]

This theorem makes the relation between two points of view on torsion. The
first depends on cell structure and homology basis (and as indicated above, has
a smaller indeterminacy). The other one depends only on the homology modules (and
in particular does not require any choice of basis), and is defined up to multiplication.
by a unit in $\mathbb{F}[t^{\pm 1}]$. For more details, see [18]. Notice that as a consequence, $\tau_{\varepsilon, \rho}(X)$ is a homotopy invariant. The following lemma will be useful later.

**Lemma 2.9** ([18]). For all $(X, \varepsilon, \rho)$ such that $\varepsilon$ is non-trivial, $H^0_{\varepsilon, \rho}(X; \mathbb{F}[t^{\pm 1}])$ is a torsion $\mathbb{F}[t^{\pm 1}]$-module.

2.4. Fox calculus. The first general definition of twisted Alexander polynomials is due to Wada [33], for finitely presented groups endowed with an abelianization and a linear representation. Its construction involves only Fox calculus.

Suppose that $\pi$ has a presentation

$$\pi = \langle x_1, \ldots, x_m | r_1, \ldots, r_n \rangle.$$ 

The representation $\varepsilon \otimes \rho$ induces a ring homomorphism

$$\mathbb{Z}[\pi] \rightarrow \mathcal{M}_r(\mathbb{F}[t^{\pm 1}])$$

$$\gamma \mapsto t^\varepsilon(\gamma) \rho(\gamma).$$

Let $F_m$ be the free group generated by $x_1, \ldots, x_m$. Set

$$\Phi : \mathbb{Z}[F_m] \rightarrow \mathbb{Z}[\pi] \rightarrow \mathcal{M}_r(\mathbb{F}[t^{\pm 1}]).$$

Following [33, Lemma 1], there exists some $i$ such that $\Phi(x_i - 1)$ has a non-zero determinant. Let $p_i : (\lambda')^m \rightarrow (\lambda')^{m-1}$ be the projection with kernel the $i$-th copy of $\lambda'$. Consider the $(nr \times mr)$-matrix

$$\Upsilon = \left[ \Phi \left( \frac{\partial r_k}{\partial x_l} \right) \right].$$

and define

$$Q_i = \begin{cases} \gcd \{ (r(m-1) \times r(m-1)) - \text{minors of } (p_i \Upsilon) \} & \text{if } m \geq n \\ 1 & \text{otherwise} \end{cases}$$

Wada defines the twisted Alexander polynomial of $(\pi; \varepsilon, \rho)$ as $Q_i / \det(\Phi(x_i - 1))$. In fact, one has the following result.

**Theorem 2.10** ([18, 33]). Let $X$ be a finite CW-complex. If $H^0_{\varepsilon, \rho}(X; \mathbb{F}[t^{\pm 1}])$ is torsion, then

$$\Delta_{\varepsilon, \rho}(X) = \frac{Q_i}{\det(\Phi(x_i - 1))}.$$ 

3. Duality theorem and intersection forms

In this section, we recall duality theorem for torsion, in our context. It was due to Franz [15] and Milnor [25], and it can be found in [18] for twisted torsion by unitary representations.

$X$ is now a compact smooth 4-manifold, with boundary $\partial X$, possibly empty. By Whitehead’s theorem, $X$ has a canonical pl-structure, unique up to ambient isotopy. In fact, any two pl-triangulations have a common linear subdivision which is pl. We endow $X$ with the CW-decomposition induced by one of these. Since $X$ is compact, the CW-complex is finite.

**Definition 3.1.** $(X, \varepsilon, \rho)$ is unitary if $\rho : \pi_1(X) \rightarrow \text{GL}(V)$ is unitary, where $V$ is finite dimensional $\mathbb{F}$-vector space, for $\mathbb{F}$ a subfield of $\mathbb{C}$ closed under conjugation.
We denote by \( \cdot : F[t^{\pm 1}] \rightarrow F[t^{\pm 1}] \) the involution induced by the complex conjugation, and \( <,> \) the positive hermitian form on \( V \). Note that \( \tau \) extends to \( F(t) \) in a natural way, so both will be denoted the same way.

**Definition 3.2.** The intersection form of a unitary \((X, \varepsilon, \rho)\) is the sesquilinear form

\[
\varphi^{\varepsilon,\rho} : H_2^\varepsilon,\rho(X, F[t^{\pm 1}]) \times H_2^\varepsilon,\rho(X, F[t^{\pm 1}]) \rightarrow F[t^{\pm 1}]
\]

\[
(f \otimes v \otimes c, g \otimes w \otimes c') \mapsto \sum_{\alpha \in \pi}(c \cdot \alpha c')f\bar{\varepsilon}(\alpha) < v\alpha, w >,
\]

where \( (\cdot) \) denotes the algebraic intersection number.

Since for each \( c \) and \( c' \), all but a finite number of terms are zero, \( \varphi^{\varepsilon,\rho} \) takes values in \( F[t^{\pm 1}] \).

Let us fix the following convention for manifolds with boundary. We say that \((Y, \tilde{\varepsilon}, \tilde{\rho})\) is the boundary of \((X, \varepsilon, \rho)\) if \( \partial X = Y \) as manifolds and the two following diagrams commute (with \( \varepsilon \) and \( \rho \) respectively), where \( i^\ast \) is induced by the inclusion:

\[
\begin{array}{ccc}
\pi_1(Y) & \xrightarrow{i^\ast} & \pi_1(X) \\
\varepsilon,\rho & \searrow & \\
\mathbb{Z} & \nearrow & \varepsilon,\rho
\end{array}
\]

From now on, \( \tilde{\varepsilon} \) and \( \tilde{\rho} \) are simply denoted by \( \varepsilon \) and \( \rho \).

**Theorem 3.3.** For any unitary \((X^4, \varepsilon, \rho)\) such that \( X \) has the homotopy type of a 2-dimensional complex and \( C^\varepsilon,\rho_\ast(X, \partial X; F(t)) \) is acyclic, the following holds

\[
\tau_{\varepsilon,\rho}(\partial X) = \tau_{\varepsilon,\rho}(X) \cdot \det(\varphi^{\varepsilon,\rho}).
\]

As mentioned before, this is a specialization of the duality theorem. The arguments used in the proof are standard, and can be found in particular in [18]. For the reader’s convenience, we briefly recall it.

**Proof.** Note that \( \partial X \) inherits the structure of a pl-manifold from \( X \) and the triangulations of \( X \) can be used to define \( C^\varepsilon,\rho_\ast(X, \partial X; F(t)) \), which is generated by cells (instead of simples). Along the lines of this proof, we will assume that the cell complexes have coefficients in \( F(t) \). Consider the pairings:

\[
C_i^\varepsilon,\rho(X) \times C_{4-i}^\varepsilon,\rho(X, \partial X) \rightarrow F(t), \quad i = 1, \ldots, 4,
\]

which induce \( F(t) \)-isomorphisms

\[
C_{4-i}^\varepsilon,\rho(X, \partial X) \rightarrow \underline{\text{Hom}}_{F(t)}(C_i^\varepsilon,\rho(X); F(t)).
\]

These isomorphisms take the differential of \( C^\varepsilon,\rho_\ast(X, \partial X) \) to the dual of the differential of \( C^\varepsilon,\rho_\ast(X) \) and induce Poincare duality isomorphisms

\[
H_{4-i}^\varepsilon,\rho(X, \partial X) \cong H_i^\varepsilon,\rho(X).
\]

The universal coefficient theorem applied to the chain complex \( C^\varepsilon,\rho_\ast(X, \partial X) \) implies that evaluation induces an isomorphism

\[
H_{4-i}^\varepsilon,\rho(X) \cong \underline{\text{Hom}}_{F(t)}(H_i^\varepsilon,\rho(X, \partial X), F(t)),
\]

and the inner product above induces the non-singular pairing
\[ H^\varepsilon_1(X) \times H^\varepsilon_2(X, \partial X) \longrightarrow \mathbb{F}(t). \]
For fixed bases of \( H^\varepsilon_1(X) \), choose the dual bases for \( H^\varepsilon_2(X, \partial X) \). We get \( \tau(e, \rho)(X, \partial X) \cdot \tau(e, \rho)(X) = 1 \), and by Lemma 2.2
\[ \tau(e, \rho)(X, \partial X) \tau(e, \rho)(\partial X) \tau(\mathcal{H}) = \tau(e, \rho)(X), \]
where \( \tau(\mathcal{H}) \) is the torsion of the long exact sequence in homology of the pair \((X, \partial X)\). Since \( C^\varepsilon_1(\partial X; \mathbb{F}(t)) \) is acyclic, \( \tau(\mathcal{H}) \) is the alternating product of the determinant of maps induced by inclusion. Moreover, since \( X \) has the homotopy type of a 2-complex, we have, from duality above
\[ H^\varepsilon_0(X) = H^\varepsilon_1(X, \partial X) = 0. \]
This implies in particular that the maps \( H^\varepsilon_1(X) \to H^\varepsilon_1(X, \partial X) \) and \( H^\varepsilon_2(X) \to H^\varepsilon_2(X, \partial X) \) are zero. To conclude the proof, consider the diagram:
\[
\begin{array}{ccc}
H^\varepsilon_2(X) \times H^\varepsilon_2(X) & \xrightarrow{\text{Id} \times i_v} & H^\varepsilon_2(X) \times H^\varepsilon_2(X, \partial X) \\
\phi^\varepsilon_\rho \otimes \mathbb{F}(t) & & \mathbb{F}(t)
\end{array}
\]
The diagonal map on the right is the unimodular map considered above. It follows that the matrices for the inclusion \( H^\varepsilon_2(X) \to H^\varepsilon_2(X, \partial X) \) and \( \phi^\varepsilon_\rho \otimes \mathbb{F}(t) \) are conjugated. Hence, \( \tau(\mathcal{H}) = 1/\det(\phi^\varepsilon_\rho) \). Note that since \( X \) has the homotopy type of a 2-complex, \( H^\varepsilon_2(X; \mathbb{F}[t^{\pm 1}]) \) is free and a matrix for \( \phi^\varepsilon_\rho \) is also a matrix for \( \phi^\varepsilon_\rho \otimes \mathbb{F}(t) \). \( \square \)

4. Knots and Links

In this section, we briefly collect results on the torsion applied to link complements and Alexander polynomials. Let \( L \) be an oriented link in \( S^3 \) with \( \mu \) components, and \( E \) be the exterior of \( L \). Denote the homology classes of the meridians of the link components by \( \nu_i \). Note that
\[ H_1(E) = \bigoplus_{i=1}^\mu \mathbb{Z}\nu_i. \]
Let \( q_1, \ldots, q_\mu \) be integers with \( \gcd(q_1, \ldots, q_\mu) = 1 \), and define
\[ e : H_1(E) \longrightarrow \mathbb{Z} = \langle t \rangle \]
\[ \nu_i \longmapsto t^{q_i}. \]
Since \( \gcd(q_1, \ldots, q_\mu) = 1 \), the associated infinite cyclic covering is connected. Let \( \rho : \pi_1(E) \longrightarrow \text{GL}(V) \) for some finite dimensional \( V \) over a field \( \mathbb{F} \).

**Definition 4.1.** The twisted torsion of \((L, e, \rho)\) is \( \tau_{e, \rho}(L) = \tau_{e, \rho}(E) \). Similarly, the twisted Alexander polynomials are \( \Delta^i_{e, \rho}(L) = \Delta^i_{e, \rho}(E) \), and denote \( \Delta_{e, \rho}(L) = \Delta^1_{e, \rho}(L)/\Delta^0_{e, \rho}(L) \).

For knots, Cha [11] also considered the case where \( \mathbb{F} \) is not a field but any Noetherian factorization domain. If \( \rho \) is the trivial representation and \( q_i = 1 \) for all \( i \), \( \Delta^1_{e, \rho}(L) = \Delta_L \) is the classical Alexander polynomial.

A CW-structure on \( E \) can be given with one 0-cell, \( n \) 1-cells and \((n-1) \) 2-cells. This can correspond to a Wirtinger presentation \( \pi_1(E) = \langle x_1, \ldots, x_n \mid \)
4.2 classes \( \nu \) of the irreducible components. Hence that bounds a fiber of boundary of a tubular neighborhood polynomials can be defined as \( \Delta_{C_{\mathcal{B}}} \). It is worth mentioning that Cha [denote \( \Delta \)]

Definition 5.1. The torsion of \((C, \varepsilon, \rho)\) is \( \tau_{C, \varepsilon, \rho}(L) = \Delta_{C, \varepsilon, \rho}(L) \). Similarly, Alexander polynomials can be defined as \( \Delta_{C, \varepsilon, \rho}^1(C) = \Delta_{C, \varepsilon, \rho}^1(X) \). As in the case of links, we will denote \( \Delta_{C, \varepsilon, \rho}^1(C)/\Delta_{C, \varepsilon, \rho}^0(C) \) simply by \( \Delta_{C, \varepsilon, \rho}(C) \).

Around 1930’s Zariski and Van-Kampen developed a method to compute the fundamental group of \( X \). Refinements of this algorithm were constructed later...
mainly by Chisini and Moishezon. Finally, Libgober described the homotopy type of $X$ as follows.

Consider a generic linear projection $C^2 \rightarrow \mathbb{C}$, i.e. such that:

1. there are no vertical asymptotes,
2. the fibers are transversal to $C$ except for a finite number of them which are either simple tangents to a point of $C$ or lines through a singular point of $C$ transversal to its tangent cone.

Let $P$ be the (finite) set of critical values of the projection. The braid monodromy of $C$ is the homomorphism

$$\vartheta : \pi_1(C - P) \rightarrow B_d,$$

where $B_d$ denotes the braid group, viewed as the mapping class group of $C^2$. Fix an arbitrary basis $\{ \alpha_i \}_{1 \leq i \leq n}$ of $\pi_1(C - P)$, and $\gamma_1, \ldots, \gamma_d$ a basis of the (free) fundamental group of a generic fiber.

In the following theorem, the presentation of $\pi_1(X)$ is due to [34, 16]. Libgober [22] used this presentation to describe the homotopy type of $X$. We recall this well-known result in order to set notation for future reference.

**Theorem 5.2.** Let $(\sigma_1, \ldots, \sigma_n) = (\vartheta(\alpha_1), \ldots, \vartheta(\alpha_n))$. The two-dimensional complex associated to the following presentation of $\pi_1(X)$:

\[
\langle \gamma_1, \ldots, \gamma_d \mid \sigma_i(\gamma_j) = \gamma_j, \quad i = 1, \ldots, n, \quad j = 1, \ldots, d \rangle,
\]

has the homotopy type of $X$.

Note that Theorem 5.2 provides in particular a presentation of the fundamental group and can be useful to construct $\rho$ explicitly.

**Proposition 5.3.** For any algebraic curve $C$ in $C^2$, one has

$$\tau_{\varepsilon, \rho}(C) = \Delta_{\varepsilon, \rho}(C).$$

In particular, if $\rho$ is the trivial representation and $q_i = 1$, then

$$(t - 1)\tau_{\varepsilon, \rho}(C) = \Delta_C.$$
Moreover, if \( \rho \) is trivial,\[
\Delta_\varepsilon^0(C) = \gcd(t^{\rho} - 1) = t - 1.
\]

In some cases, one can assure that \( \Delta_{\varepsilon, \rho}(C) \) is actually a polynomial.

**Proposition 5.4.** If \( C \) is not irreducible, then \( \Delta_{\varepsilon, \rho}(C) \) is a polynomial.

**Proof.** Let us consider ab : \( H_1(\mathbb{C}^2 \setminus C) \to \mathbb{Z}^r \) the abelianization morphism. Analogously to subsection 2.4, the several-variable invariant \( \Delta_{ab, \rho}(C) \) can be defined as \( \frac{Q_i(t_1, \ldots, t_r)}{P_i(t_1, \ldots, t_r)} \), where \( Q_i \) is the gcd of the minors of maximal order of \( p_i Y_{ab, \rho} \) and \( P_i = \det(\text{Id} - \rho(\nu_i)t_i) \). By [33] \( \Delta_{ab, \rho}(C) \) is a polynomial in the variables \( (t_1, \ldots, t_r) \).

Consider also \( \Delta_{\varepsilon, \rho}(C) = \frac{\tilde{Q}(t)}{\tilde{P}(t)} \), where \( \tilde{Q} \) is the gcd of the minors of maximal order of \( p_i Y_{\varepsilon, \rho} \) and \( \tilde{P}_i = \det(\text{Id} - \rho(\nu_i)t_i) \). Note that \( P_i(t^{\nu_1}, \ldots, t^{\nu_r}) = \tilde{P}_i(t) \) and \( P_i(t_1, \ldots, t_r) \) divides every minor in \( p_i Y_{ab, \rho} \). Therefore \( P_i(t^{\nu_1}, \ldots, t^{\nu_r}) = \tilde{P}_i(t) \) divides every minor of \( p_i Y_{\varepsilon, \rho} \). This means that \( \tilde{P}_i(t) \) also divides \( \tilde{Q}_i(t) \), and thus \( \Delta_{\varepsilon, \rho}(C) \) is also a polynomial. \( \square \)

**5.2. Relation with local polynomials.** Suppose that we are given \( (C, \varepsilon, \rho) \). Let \( S_1^3, \ldots, S_s^3 \) be sufficiently small 3-spheres around the singular points \( \{P_1, \ldots, P_s\} \) of \( C \). Denote by \( L_k = C \cap S_k^3 \) the link of the singularity at \( P_k \), and by \( E_k \) be the link exterior. Also denote by \( \pi^k = \pi_1(S_k^3 - L_k) \) the local fundamental groups at \( P_k \). The inclusion maps \( i_k : \pi^k \to \pi_1(X) \) and \( (\varepsilon, \rho) \) induce morphisms

\[
\varepsilon_k : \pi^k \to \mathbb{Z}
\]

and

\[
\rho_k : \pi^k \to \text{GL}(V),
\]

for any \( k = 1, \ldots, s \).

**Definition 5.5.** For all \( k = 1, \ldots, s \), let the local torsions be defined as

\[
\tau_k = \tau_{\varepsilon_k, \rho_k}(L_k).
\]

Note that an explicit description of the maps \( \pi^k \to \pi \) can be obtained via the braid monodromy of a generic projection of the curve (accurate packages have been developed to compute braid monodromies of curves with equations over the rationals by Bessis [8] and Carmona [10]). It can be computed as follows. The relators given in Theorem 5.2(1) can be decomposed as

\[
\sigma_k(\gamma_j) = \omega_{k,j} \sigma_k(\gamma_{k,j}) \omega_{k,j}^{-1} = \gamma_j,
\]

where \( \gamma_{k,j} \) and \( \gamma_j \) are meridians of the same irreducible component of \( C \), \( \gamma_{k,j} \) is in \( S_k^3 \setminus L_k \) and \( \sigma_k \) only depends on the local type of the singularity \( P_k \). Note that not all \( j \)'s are valid for any \( k \), only those for which the component of \( \gamma_j \) passes through \( P_k \). We will denote such set of subindices by \( \Sigma_k \). Under such conditions \( \pi^k \) can be described as

\[
\pi^k = \langle \gamma_{k,j} \mid \sigma_k(\gamma_{k,j}) = \gamma_{k,j}, j \in \Sigma_k \rangle
\]

and \( \rho_k(\gamma_{k,j}) = \omega_{k,j}^{-1} \gamma_j \omega_{k,j} \), for any \( j \in \Sigma_k \).

Let \( \varphi_{\varepsilon, \rho}(C) \) be the intersection form of \( (X, \varepsilon, \rho) \) defined as in 3.2. Theorem 1.1, stated in the introduction, can be written as follows.
Theorem 5.6. Let \((C, \varepsilon, \rho)\) be unitary and suppose that the local representations \(\rho_k\) are acyclic. Then
\[
\left( \prod_{i=1}^{r} \det(\Id - \rho(\nu_i)t^{q_i})^{s_i-\chi(C_i)} \right) \cdot \prod_{k=1}^{s} \tau_k = \tau_{\varepsilon, \rho}(C) \cdot \tau_{\varepsilon, \rho}(C) \cdot \det \varphi_{s, \rho}(C),
\]
where \(\nu_i\) is the homology class of a meridian around the irreducible component \(C_k\) and \(s_i = \#\text{Sing}(C) \cap C_k\).

Remark 5.7.

1. From the discussion above, since two meridians of the same irreducible component are conjugated, \(\det(\Id - \rho(\gamma_i)t^{q_i})\) only depends on the homology class of \(\gamma_i\), say \(\nu_i\). By abuse of notation, we simply write \(\det(\Id - \rho(\nu_i)t^{q_i})\) for this element of \(F[t^{\pm 1}]\).

2. Note that in the case \(F = C\), one can canonically obtain a unitary representation from any given representation \(\rho\) as follows. Let \(\rho\) be a representation of \(\pi_1(X)\), if \(\gamma_i\) and \(\gamma_j\) are meridians of the same irreducible component of the curve \(C\), then \(\det(\rho(\gamma_i)) = \det(\rho(\gamma_j))\). In other words, \(\rho\) induces a morphism in homology \(\det: H_1(X) \to H_1(C)\), given by \(\det(\rho(\gamma_i)) = \det(\rho(\gamma_j))\), where \(\gamma_i\) is the homology class of \(\gamma_j\). Therefore, \(\det(\rho(\gamma_i)) = \det(\rho(\gamma_j))\) in Theorem 5.2(1) and hence \(\det(\rho(\gamma_i)) = 1\).

Corollary 5.8. If \(\varphi^t(C)\) is the intersection form with twisted coefficients in \(Q[t^{\pm 1}]\), then
\[
(t-1)^{2-\chi(C)} \prod_{k} \Delta_{L_k} = \Delta_{C} \cdot \Delta_{\varepsilon} \cdot \det \varphi^t(C).
\]

Proof. An immediate consequence of Propositions 4.2, 5.3, and Theorem 5.6 considering \(\rho\) trivial and \(q_i = 1, i = 1, \ldots, r\); in addition to the obvious relation \(\sum_{i=1}^{r} (s_i - \chi(C_i)) = s - \chi(C)\) \(\square\)

Proof. Theorem 5.6.

Let \((M, \varepsilon, \rho)\) be the boundary of the curve exterior and let \(F = C - \cup_k (C \cap B^3_k)\) be the (abstract) surface obtained by removing disks \(D_1 \cup \cdots \cup D_{n_k}\) around each singular point \(F_k\) of \(C\). Let \(N = F \times S^1\). The boundary of \(N\) is a union of disjoint tori \(T^k_i \cup \cdots \cup T^k_{n_k}\) for \(k = 1, \ldots, s\). From a plumbing description of a tubular neighborhood of the curve, one can show that \(M\) is obtained by gluing the link exteriors \(E_k\) with \(N = F \times S^1\), along the tori \(T^k_i\) for \(i = 1, \ldots, n_k\):

\[
M = N \cup_{\mathcal{T}_i} (\prod_k S^3_k - L_k).
\]

The gluing map sends a longitude of \(L_k\) to the restriction of a section in \(N\), and a meridian of \(L_k\) to a fiber in \(N\). The inclusion maps induce triples \((N, \varepsilon, \rho)\), \((S^3_k - L_k, \varepsilon_k, \rho_k)\), and \((T^k_i, \varepsilon_k, \rho_k)\) (with \(k = 1, \ldots, s\), \(i = 1, \ldots, n_k\)). By multiplicativity, \(\tau_{\varepsilon, \rho}(N)\) is the product of the torsion of the connected components of \(N\). Let us compute \((F_t \times S^1, \varepsilon, \rho)\) where \(F_t\) is a connected component of \(F\), corresponding to \(C_k\). Note that \(H_1(F_t \times S^1) = H_1(F_t) \oplus \Z\varepsilon_t\) and \(\varepsilon(\nu_t) = q_t\). Hence by Lemma 2.9, \(H^*_{\varepsilon, \rho}(F_t \times S^1)\) is torsion over \(F[t^{\pm 1}]\). If \(F_t\) is a disk, then \(\varepsilon(\nu_t) = q_t\) and \(\tau_{\varepsilon, \rho}(F_t \times S^1) = 1\). If \(F_t\) is not a disk, then \(\tau_{\varepsilon, \rho}(F_t \times S^1)\) can be presented by \(b_1(F_t) + 1\) generators \(a_1, \ldots, a_{b_1(F_t)}\), \(\gamma_t\) (such that \(\bar{\gamma_t} = \nu_t\)) with the relations \(a_k\gamma_t = \gamma_t a_k\) for
all \( k = 1, \ldots, b_1(F_\ell) \). Taking the matrix of Fox derivatives of the relations and tensoring with \( \mathbb{F}[t^{\pm 1}] \otimes V \) one obtains a \( b_1(F_\ell) \times (b_1(F_\ell) + 1) \) matrix, with entries in \( M_{m \times m}(\mathbb{F}[t^{\pm 1}]) \) (\( m = \dim V \)) describing the differential
\[
\partial_2 : C_2(F_\ell \times S^1; \mathbb{F}[t^{\pm 1}]) \longrightarrow C_1(F_\ell \times S^1; \mathbb{F}[t^{\pm 1}]).
\]
The \( i \)-th row of this matrix has \( \text{Id} - \rho(\gamma_i)t^{\varepsilon} \) in the \( \ell \)-th column and \( \text{Id} - \rho(a_i)t^{\varepsilon(a_i)} \) in the last column. The zero chains are \( C_0(F_\ell \times S^1; \mathbb{F}[t^{\pm 1}]) = \mathbb{F}[t^{\pm 1}] \otimes V \) and \( \partial_1 \) is the column matrix with \( i \)-th entry \( \rho(a_i)t^{\varepsilon(a_i)} \text{Id} \) for \( i \leq b_1(F_\ell) \) and last entry \( \rho(\gamma_1)t^{\varepsilon} - \text{Id} \). Dropping the last column of the matrix for \( \partial_2 \), one obtains a \( b_1(F_\ell) \times b_1(F_\ell) \) matrix with determinant \( \det(\text{Id} - \rho(\gamma_i)t^{\varepsilon})^{b_1(F_\ell)} \). Note that this determinant only depends on the conjugacy class \( \nu \) of \( \gamma_1 \), and hence, it does not depend on the chosen meridian of \( C_\ell \). Therefore, in the future we will simply write \( \det(\text{Id} - \rho(\nu)t^{\varepsilon}) \).

Also note that the chain complex \( C_{\ast, \rho}^C(F_\ell \times S^1; \mathbb{F}(t)) \) is acyclic since \( q_\ell \neq 0 \). From Theorem 2.10, one obtains \( \tau_{\varepsilon, \rho}(F_\ell \times S^1) = \det(\text{Id} - \rho(\nu)t^{\varepsilon}) - \chi(F_\ell) \). Note that \( \chi(F_\ell) = \chi(C_\ell) - s_\ell \). By additivity of Euler characteristic, one has
\[
\tau_{\varepsilon, \rho}(N) = \prod_{\ell=1}^r \det(\text{Id} - \rho(\nu_\ell)t^{\varepsilon})^{s_\ell - \chi(C_\ell)}.
\]
Also note that, since \( \pi_1(T^k_\ell) \) is Abelian, one has
\[
\prod_{k=1}^s \prod_{\ell=1}^{n_k} \tau_{\varepsilon, \rho_k}(T^k_\ell) = 1.
\]
Hence one has the following Mayer-Vietoris sequence with coefficients in \( \mathbb{F}(t) \):
\[
0 \longrightarrow \oplus_{i,k} C_{\ast, \rho_k}^C(T^k_\ell) \longrightarrow \oplus_k \left( C_{\ast, \rho_k}^C(S^3_\ell - L_k) \right) \oplus C_{\ast, \rho}^C(N) \longrightarrow C_{\ast, \rho}^C(M) \longrightarrow 0.
\]
By hypothesis the chain complex \( \oplus_k C_{\ast, \rho_k}^C(S^3_\ell - L_k; \mathbb{F}(t)) \) is acyclic. Since \( C_{\ast, \rho}^C(N; \mathbb{F}(t)) \) and \( \oplus_{i,k} C_{\ast, \rho}^C(T^k_\ell) \) are also by computations above, then \( C_{\ast, \rho}^C(M) \) is acyclic. From Lemma 2.2, we obtain
\[
(\prod_k \tau_k) \cdot \tau_{\varepsilon, \rho}(N) = (\prod_{k,i} \tau_{\varepsilon, \rho_k}(T^k_\ell)) \cdot \tau_{\varepsilon, \rho}(M).
\]
Hence,
\[
\tau_{\varepsilon, \rho}(M) = \left( \prod_{\ell=1}^r \det(\text{Id} - \rho(\nu_\ell)t^{\varepsilon})^{s_\ell - \chi(C_\ell)} \right) \cdot \prod_k \tau_k.
\]
By Theorem 3.3, we obtain the result. 

5.3. Twisted Alexander Polynomials and Characteristic Varieties. Let \( \tilde{\xi} = (\xi_1, ..., \xi_r) \) be in the zero set of the first characteristic variety \( V_1(X) \subset (\mathbb{C}^*)^r \) associated with an affine curve \( C \subset \mathbb{C}^2 \) with \( r \) irreducible components \( C_1, C_2, ..., C_r \). In other words, \( \xi \) is in the zero set of the first fitting ideal \( F_1(X) \) of \( H_1(X_{\text{ab}}) \) as a \( \mathbb{C}[H_1(X)] \)-module (where \( X_{\text{ab}} \) represents the universal abelian cover of \( X \)). Consider the representation \( \rho : \pi_1(X) \to \mathbb{C}^* = \text{GL}(1, \mathbb{C}) \) given by \( \rho_\ell(\gamma_i) = \xi_i \), for any \( \gamma_i \) meridian of the irreducible component \( C_i \) and \( \varepsilon : H_1(X) \to \mathbb{Z} \) such that \( \varepsilon(\nu_\ell) = 1 \). Hence any polynomial \( p(t_1, ..., t_r) \in F_1(X) \) satisfies \( p(\tilde{\xi}) = 0 \). Note that \( \Delta_{\varepsilon, \rho_\ell}(C) = \gcd\{p(\xi_1, t, ..., \xi_r) \mid p \in F_1(X)\} \). Hence, either \( p(\xi_1, t, ..., \xi_r) = 0 \) for all \( p \in F_1(X) \) or \( \Delta_{\varepsilon, \rho_\ell}(C) \) contains \( t - 1 \) as a factor, since \( t = 1 \) is a root of \( p(\xi_1, t, ..., \xi_r) \). Therefore, \( \tilde{\xi} \in V_1(X) \) if and only if \( (t - 1) \) divides \( \Delta_{\varepsilon, \rho_\ell}(C) \).
6. Examples

6.1. A Zariski pair. Consider the space $\mathcal{M}$ of sextics with the following combinatorics:

(1) $C$ is a union of a smooth conic $C_2$ and a quartic $C_4$.
(2) $\text{Sing}(C_4) = \{P, Q\}$ where $Q$ is a cusp of type $A_4$ and $P$ is a node of type $A_1$.
(3) $C_2 \cap C_4 = \{Q, R\}$ where $Q$ is a $D_7$ on $C$ and $R$ is a $A_{11}$ on $C$.

As shown in [3], $\mathcal{M}$ consists of two irreducible components. One can add a transversal line and calculate the fundamental groups of representatives $C^{(1)}$ and $C^{(2)}$ in each component. One has the following:

$$\pi_1(C^2 \setminus C^{(1)}) = \langle e_1, e_2 : [e_2, e_1^2] = 1, (e_1 e_2)^2 = (e_2 e_1)^2, [e_1, e_2^2] = 1 \rangle$$
$$\pi_1(C^2 \setminus C^{(2)}) = \langle e_1, e_2 : [e_2, e_1^2] = 1, (e_1 e_2)^2 = (e_2 e_1)^2 \rangle.$$

Assuming $\varepsilon : \mathbb{Z}^2 \to \mathbb{Z}$, $\varepsilon(1,0) = \varepsilon(0,1) = 1$ and taking, for instance, the rank 1 representation $\rho(e_1) = 1$, $\rho(e_2) = -1$ (which is unique up to equivalence) one obtains:

$$\Delta_{\varepsilon, \rho}(C^{(1)}) = 1, \quad \Delta_{\varepsilon, \rho}(C^{(2)}) = t + 1.$$  

Also, note that the classical Alexander polynomial is trivial for both curves.

6.2. Nodal degenerations. The following example illustrates how twisted Alexander polynomials can be sensitive to nodes, something that classical Alexander polynomials are not.

We say a curve $\mathcal{D}$ is of type $I$ if $\mathcal{D}$ is an irreducible plane curve of degree $d$ such that $\mathcal{D}$ has an ordinary $(d-2)$-ple point at $P$. Let $L_1$ and $L_2$ be lines through $P$ such that either $L_i$ is tangent to a smooth point $P_i \in \mathcal{D}$ or $L_i$ passes through a double point $P_i \neq P$ of type $A_{2r}$. Let us denote $\mathcal{C} = L_1 + L_2 + \mathcal{D}$. Assume that $\mathcal{D}$ has only nodes as singular points apart from $P$. According to [6, Theorem 1], $\mathcal{D}$ is rational if and only if there exists a dihedral cover $\mathbb{D}_{2n}$ ramified along $2(L_1 + L_2) + n\mathcal{D}$ for any $n \geq 3$. In fact, according to [7, Corollary 2] this implies that $\mathcal{D}$ is rational if and only if the fundamental group of $\mathbb{P}^2 \setminus (L_1 \cup L_2 \cup \mathcal{D})$ admits $\mathbb{Z}_2 * \mathbb{Z}_2$ as a quotient. Moreover (see [6, Proposition 6.1]), there exist nodal degenerations $\mathcal{D}_t \to \mathcal{D}_0$ to a rational $\mathcal{D}_0$ of type $I$ using (non-rational) curves $\mathcal{D}_t \ (t > 0)$ of type $I$ with Abelian fundamental groups. A presentation for the fundamental group of $\mathcal{C}_0 = L_1 + L_2 + \mathcal{D}_0$ is given as follows:

$$G(\mathcal{C}_0) = \langle \ell, x_1, x_2 \mid [x_1, x_2] = 1, \ell^{-1}x_1 \ell = x_2, \ell^{-1}x_2 \ell = x_1 \rangle.$$  

Considering $\varepsilon$ the usual morphism $\varepsilon(\nu_t) = 1$, and

$$\rho(\ell) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\rho(x_1) = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix},$$
$$\rho(x_2) = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$$

one obtains

$$\Delta_{\varepsilon, \rho}(\mathcal{C}_0) = t + 1.$$  

Note that $\rho(G(\mathcal{D}_0)) \cong \mathbb{Z}_2 * \mathbb{Z}_2$. 

The three non-nodal singularities of $C_t$ are $\{P, P_1, P_2\}$ and they lay on the lines $L_1$ and $L_2$, hence maybe be performing projective transformations, we can assume that $\{P, P_1, P_2\}$ and $L_1$ and $L_2$ are fixed all through the degeneration. This implies that the classical Alexander polynomial $\Delta_{C_t}$ of $C_t$ is the same for all $t \geq 0$ (since they have the same adjunction ideals, see for instance [21, Theorem 5.1]). Since $G(C_t)$ is Abelian, this implies that $\Delta_{C_1} = \Delta_{C_0} = 1$. Formula (2) shows that $C_0$ has a non-trivial twisted Alexander polynomial.

References


Departamento de Matemáticas, Universidad de Zaragoza
E-mail address: jicogo@unizar.es

Departamento de Álgebra y Geometría, Universidad de Valladolid
E-mail address: vincent_florens@yahoo.fr