

Formality of a K3 surface from a computational viewpoint

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Introduction

The rational homotopy type of a space is a simplified version of the homotopy type where all homotopy groups are tensored by $\mathbb{Q}$.

The lost information in Rational Homotopy. The case of $\mathbb{S}^n$

	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}
$\mathbb{S}^4$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	$\mathbb{Z}_5$	$\mathbb{Z}_2$
$\mathbb{S}^5$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$
$\mathbb{S}^6$	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$

Table 1: A subset of the homotopy groups of the spheres $\mathbb{S}^4, \mathbb{S}^5$ and $\mathbb{S}^6$.

Something we know:

Homotopy groups of $\mathbb{S}^k$ vanish until degree $k - 1$ (included).

$$\pi_k(\mathbb{S}^k) = \mathbb{Z}$$

π_n is commutative for $n \geq 2$

$$\pi_{2n}(\mathbb{S}^{2n}) \otimes \mathbb{Q} = \pi_{4n-1}(\mathbb{S}^{2n}) \otimes \mathbb{Q} = \mathbb{Q}$$

Despite the lost information, rational homotopy has the advantage of being constructive.

Due to Sullivan, it is known how to obtain the rational homotopy type of a topological space X , via the construction of a commutative differential graded algebra, called the Sullivan model of X . This algebra allows us to establish a categorical equivalence between homotopy types of spaces and isomorphism classes of Sullivan models:

$$\left\{ \begin{array}{l} \textit{Rational homotopy} \\ \textit{types of spaces} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \textit{isomorphism classes of} \\ \textit{minimal Sullivan models} \end{array} \right\}$$

Formality

Formality of a topological space

It is said that a topological space is formal if the rational homotopy type of a space is a *formal* consequence of its cohomology.

It means that knowing the cohomology type of the space, we know also the rational homotopy type.

Non-formal spaces

If a topological space has (rational) homotopical information that can not be deduced from its cohomology type, then it is said that the space it is not formal.

Structures we work with

Commutative Differential Graded Algebras (CDGAs)

Simplicial structures

Cochain complexes

Commutative Differential Graded Algebras (GCDAs)

Definition

A graded commutative differential algebra (or GCDA), A , over a ring $\mathfrak{R}$ is a graded $\mathfrak{R}$ -algebra, $A = \bigoplus_{i=0}^{\infty} A^i$, together with an $\mathfrak{R}$ -linear map $d_A : A \rightarrow A$ that satisfies the following conditions:

$$xy \in A^{i+j}, \forall x \in A^i, \forall y \in A^j$$

$$xy = (-1)^{ij}yx, \forall x \in A^i, \forall y \in A^j$$

$$d_A^2 = 0$$

$$d_A(x) \in A^{i+1}, \forall x \in A^i$$

$$d_A(xy) = d_A(x)y + (-1)^i x d_A(y), \forall x \in A^i, \forall y \in A$$

Note

From now on, we will assume that $\mathfrak{R} = \mathbb{Q}$

Definition

A GCDA is said to be i -minimal (in the sense of Sullivan) if satisfies the following conditions:

- it is freely generated as a graded commutative algebra

- it is generated by a collection of elements $\{a_l\}$, with $l \in J$ (for a well ordered index set J) which satisfy that

$$|a_l| \leq |a_s| \leq i, \text{ if } l \leq s$$

- The differential of a generator a_s is expressed in terms of the generators a_l 's with $l < s$.

Definition

A morphism of algebras, $\phi : A \rightarrow B$, it is said to be a morphism of differential algebras if it preserves the degree and commutes with the differential.

Definition

Two differential algebras, A, B , are said to be i -quasi-isomorphic if there exists a morphism of differential algebras such that:

$\phi^* : H^j(A) \rightarrow H^j(B)$ is an isomorphism for every $j \leq i$

$\phi^* : H^{i+1}(A) \rightarrow H^{i+1}(B)$ is injective.

Definition

An i -minimal model of a GCDA, (A, d_A) , is an i -minimal differential graded algebra, (M, d_M) , together with an i -quasi-isomorphism,

$$\phi : M \rightarrow A$$

Definition

A minimal model of a GCDA, (A, d_A) , is a differential graded algebra (M, d_M) and a quasi-isomorphism ϕ that satisfy:

(M, d_M) is i -minimal $\forall i \geq 0$

ϕ is i -quasi-isomorphic $\forall i \geq 0$

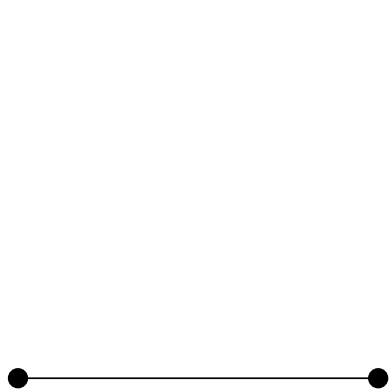
Definition

A GCDA (A, d_A) is said to be connected if $A^0 = \mathbb{Q}$

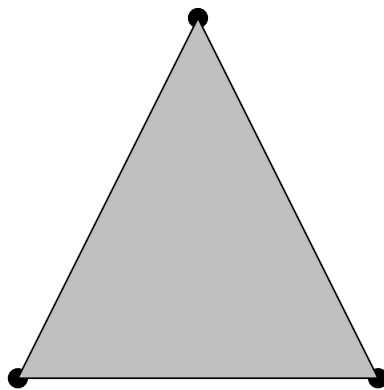
Theorem [Morgan 1978; Sullivan 1977]

Every connected differential graded algebra A has a minimal model, unique up to isomorphism.

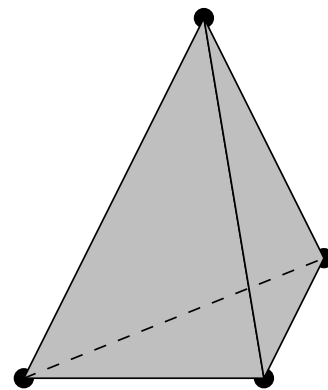
Building blocks for simplicial complexes



(a) 1-simplex

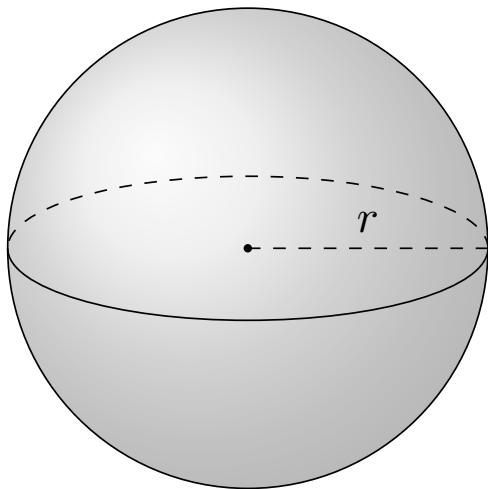


(b) 2-simplex

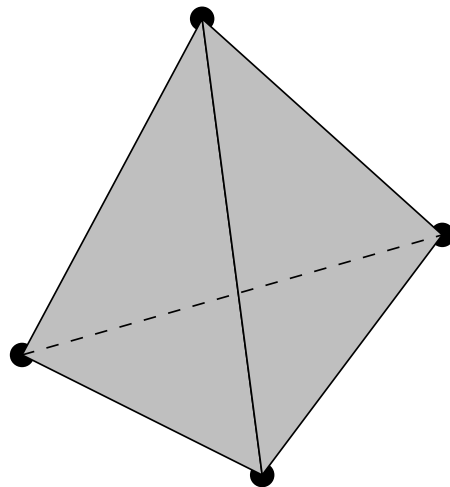


(c) 3-simplex

Simplicial complexes



(a) 2-Sphere



(b) 3-simplex

Simplicial objects

Definition

A simplicial object K with values in a category $\mathfrak{C}$ is a sequence $\{K_n\}_{n \geq 0}$ of objects in $\mathfrak{C}$, together with $\mathfrak{C}$ -morphisms:

$$\delta_i : K_{n+1} \rightarrow K_n, \quad 0 \leq i \leq n+1, \quad s_j : K_n \rightarrow K_{n+1}, \quad 0 \leq j \leq n$$

that satisfy the identities:

$$\delta_i \delta_j = \delta_{j-1} \delta_i, \quad i < j$$

$$s_i s_j = s_{j+1} s_i, \quad i \leq j$$

$$\delta_i s_j = \begin{cases} s_{j-1} \delta_i, & i < j \\ id, & i = j, j+1 \\ s_j \delta_{i-1}, & i > j+1 \end{cases}$$

Simplicial objects

Definition

A simplicial set K is a simplicial object in the category of sets.

Definition

A simplicial cochain algebra A is a simplicial object in the category of cochain algebras.

Definition

A simplicial morphism $f : L \rightarrow K$ between two such simplicial objects is a sequence of $\mathfrak{C}$ -morphisms $\varphi_n : L_n \rightarrow K_n$ commuting with the δ_i and s_j .

The commutative cochain algebra $(A_{PL})_n$

Consider the free graded commutative algebra

$\wedge(t_0, \dots, t_n, y_0, \dots, y_n)$ in which the basis elements t_i have degree 0 and the basis elements y_j have degree 1. The differential is defined as:

$$d(t_i) = y_i$$

$$d(y_j) = 0$$

The simplicial commutative cochain algebra $\{(A_{PL})_n\}_{n \geq 0}$

Definition

We define the simplicial cochain algebra $A_{PL} = \{(A_{PL})_n\}_{n \geq 0}$ by:

The cochain algebra $(A_{PL})_n$ is given by:

$$(A_{PL})_n = \frac{\wedge(t_0, \dots, t_n, y_0, \dots, y_n)}{\langle \sum t_i - 1, \sum y_j \rangle}$$

The simplicial commutative cochain algebra A_{PL}

Definition (cont.)

The face and degeneracy morphisms are the unique cochain algebra morphisms:

$$\delta_i : (A_{PL})_{n+1} \rightarrow (A_{PL})_n, \quad 0 \leq i \leq n+1$$

$$s_j : (A_{PL})_n \rightarrow (A_{PL})_{n+1}, \quad 0 \leq j \leq n$$

satisfying

$$\delta_i = t_k \rightarrow \begin{cases} t_k, & k < i \\ 0, & k = i \\ t_{k-1}, & k > i \end{cases} \quad s_j = t_k \rightarrow \begin{cases} t_k, & k < j \\ t_k + t_{k+1}, & k = j \\ t_{k+1}, & k > j \end{cases}$$

The quasi-isomorphism $\oint$

If K is a simplicial set, then $A_{PL}(K)$ is a GCDA, where $C^*(K)$ is not, but the first one is constructed in order to be quasi-isomorphic to the latter:

Definition

The integration $\oint : A_{PL}(K) \rightarrow C^*(K)$ provides a natural quasi-isomorphism of cochain complexes.

$$\oint = \{\oint_n\}_{n \geq 0}$$

$$\oint_n : A_{PL}(K)_n^n \rightarrow C^n(K)$$

The Sullivan model of a space

Definition

We define the Sullivan model of a space X , as the minimal model of the graded commutative differential algebra $A_{PL}(X)$.

Structures and maps involved in the computation of the Sullivan model of $A_{PL}(K)$.

$$\begin{array}{ccccc}
 M^{p-1} & \xrightarrow{d_M^{p-1}} & M^p & \xrightarrow{d_M^p} & M^{p+1} \\
 \downarrow \varphi^{p-1} & & \downarrow \varphi^p & & \downarrow \varphi^{p+1} \\
 A_{PL}^{p-1}(K) & \xrightarrow{d_A^{p-1}} & A_{PL}^p(K) & \xrightarrow{d_A^p} & A_{PL}^{p+1}(K) \\
 \downarrow \mathfrak{f}^{p-1} & & \downarrow \mathfrak{f}^p & & \downarrow \mathfrak{f}^{p+1} \\
 C^{p-1}(K) & \xrightarrow{d_C^{p-1}} & C^p(K) & \xrightarrow{d_C^p} & C^{p+1}(K)
 \end{array}
 \qquad
 \begin{array}{c}
 H^p(M) \\
 \downarrow \varphi_*^p \cong \\
 H^p(A_{PL}(K)) \\
 \downarrow \mathfrak{f}_*^p \cong \\
 H^p(C^*(K))
 \end{array}$$

A section for the integral map (Whitney map)

$$A_{PL}^p(K) \xrightarrow{\mathfrak{f}^p} C^p(K)$$


A red curved arrow points from $C^p(K)$ back to $A_{PL}^p(K)$, labeled W^p in red.

Properties

$$\mathfrak{f}^p \circ W^p = Id_C$$

$$W^p \circ \mathfrak{f}^p \simeq Id_A$$

Formality of a K3 Surface

Number of simplices of the K3 Surface at each dimension.

	Dimension				
	0	1	2	3	4
Nº Simplices	16	120	560	720	288

The Sullivan model of a space

Definition

We define the Sullivan model of a space X , as the minimal model of the graded commutative differential algebra $A_{PL}(X)$.

Formality of a K3 Surface

The Betti numbers of the space. Omitting degree 0, the cohomology has 22 generators of degree 2 and one generator of degree 4.

	Degree				
	0	1	2	3	4
Nº of Cohomology Generators	1	0	22	0	1

Theorem

Due to Terzić, it is known that:

Let M be a closed oriented simply connected four-manifold and b_2 its second Betti number. Then:

If $b_2 > 2$ then $\dim \pi_*(M) \otimes \mathbb{Q} = \infty$ and

$$\operatorname{rk} \pi_2(M) = b_2$$

$$\operatorname{rk} \pi_3(M) = \frac{b_2(b_2 + 1)}{2} - 1,$$

$$\operatorname{rk} \pi_4(M) = \frac{b_2(b_2^2 - 4)}{3}$$

Formality of a K3 Surface

The rank of the first homotopy groups are:

$\pi_n(K3)$				
	1	2	3	4
rank	0	22	252	3520

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