

Effective computation of the rational homotopy type of a topological space via its Sullivan model

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Outline

- 1 Introduction
- 2 Graded Commutative Differential Algebras
 - Definition of a GCDA
 - The Sullivan model of a GCDA
 - An effective method to compute the minimal model
- 3 The Sullivan model of a space
 - From topology to algebra
 - Simplicial sets
 - The algebra $A_{PL}(K)$
 - The Sullivan model of $A_{PL}(K)$
 - The relation of the minimal model and the rational homotopy type of the space
- 4 My work
- 5 Applications

Introduction

The rational homotopy type of a space is a simplified version of the homotopy type where all homotopy groups are tensored by $\mathbb{Q}$.

The lost information in Rational Homotopy. The case of $\mathbb{S}^n$

	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}
$\mathbb{S}^4$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	$\mathbb{Z}_5$	$\mathbb{Z}_2$
$\mathbb{S}^5$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$
$\mathbb{S}^6$	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$

Table 1: A subset of the homotopy groups of the spheres $\mathbb{S}^4, \mathbb{S}^5$ and $\mathbb{S}^6$.

Something we know:

- Homotopy groups of $\mathbb{S}^k$ vanish until degree $k - 1$ (included).
- $\pi_k(\mathbb{S}^k) = \mathbb{Z}$
- π_n is commutative for $n \geq 2$
- $\pi_{2n}(\mathbb{S}^{2n}) \otimes \mathbb{Q} = \pi_{4n-1}(\mathbb{S}^{2n}) \otimes \mathbb{Q} = \mathbb{Q}$

Despite the lost information, rational homotopy has the advantage of being constructive.

Due to Sullivan, it is known how to obtain the rational homotopy type of a topological space X , via the construction of a commutative differential graded algebra, called the Sullivan model of X . This algebra allows us to establish a categorical equivalence between homotopy types of spaces and isomorphism classes of Sullivan models:

$$\left\{ \begin{array}{l} \textit{Rational homotopy} \\ \textit{types of spaces} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \textit{isomorphism classes of} \\ \textit{minimal Sullivan models} \end{array} \right\}$$

- 1 Introduction
- 2 Graded Commutative Differential Algebras
 - Definition of a GCDA
 - The Sullivan model of a GCDA
 - An effective method to compute the minimal model
- 3 The Sullivan model of a space
 - From topology to algebra
 - Simplicial sets
 - The algebra $A_{PL}(K)$
 - The Sullivan model of $A_{PL}(K)$
 - The relation of the minimal model and the rational homotopy type of the space
- 4 My work
- 5 Applications

Definition of a GCDA

Definition

A graded commutative differential algebra (or GCDA), A , over a ring $\mathfrak{R}$ is a graded $\mathfrak{R}$ -algebra, $A = \bigoplus_{i=0}^{\infty} A^i$, together with an $\mathfrak{R}$ -linear map $d_A : A \rightarrow A$ that satisfies the following conditions:

- $xy \in A^{i+j}, \forall x \in A^i, \forall y \in A^j$
- $xy = (-1)^{ij}yx, \forall x \in A^i, \forall y \in A^j$
- $d_A^2 = 0$
- $d_A(x) \subseteq A^{i+1}, \forall x \in A^i$
- $d_A(xy) = d_A(x)y + (-1)^i x d_A(y), \forall x \in A^i, \forall y \in A$

Note

From now on, we will assume that $\mathfrak{R} = \mathbb{Q}$

Definition of a GCDA

Example (2.1 from [Manero & Marco, 2020])

Let A be a GCDA generated by six generators of degree 1: $e_1 \dots e_6$ and one generator of degree 2: e_7 , with no relations and where the differential is given by:

$$d_A(e_1) = -e_1 \wedge e_6$$

$$d_A(e_2) = -e_2 \wedge e_6$$

$$d_A(e_3) = -e_3 \wedge e_6$$

$$d_A(e_4) = -e_5 \wedge e_6$$

$$d_A(e_5) = d_A(e_6) = d_A(e_7) = 0$$

- 1 Introduction
- 2 Graded Commutative Differential Algebras
 - Definition of a GCDA
 - The Sullivan model of a GCDA
 - An effective method to compute the minimal model
- 3 The Sullivan model of a space
 - From topology to algebra
 - Simplicial sets
 - The algebra $A_{PL}(K)$
 - The Sullivan model of $A_{PL}(K)$
 - The relation of the minimal model and the rational homotopy type of the space
- 4 My work
- 5 Applications

Definition

A GCDA is said to be i -minimal (in the sense of Sullivan) if satisfies the following conditions:

- it is freely generated as a graded commutative algebra
- it is generated by a collection of elements $\{a_l\}$, with $l \in J$ (for a well ordered index set J) which satisfy that $|a_l| \leq |a_s| \leq i$, if $l \leq s$
- The differential of a generator a_s is expressed in terms of the generators a_l 's with $l < s$.

Definition

A morphism of algebras, $\phi : A \rightarrow B$, it is said to be a morphism of differential algebras if it preserves the degree and commutes with the differential.

Definition

Two differential algebras, A, B , are said to be i -quasi-isomorphic if there exists a morphism of differential algebras such that:

- $\phi^* : H^j(A) \rightarrow H^j(B)$ is an isomorphism for every $j \leq i$
- $\phi^* : H^{i+1}(A) \rightarrow H^{i+1}(B)$ is injective.

Definition

An i -minimal model of a GCDA, (A, d_A) , is an i -minimal differential graded algebra, (M, d_M) , together with an i -quasi-isomorphism,

$$\phi : M \rightarrow A$$

Definition

A minimal model of a GCDA, (A, d_A) , is a differential graded algebra (M, d_M) and a quasi-isomorphism ϕ that satisfy:

- (M, d_M) is i -minimal $\forall i \geq 0$
- ϕ is i -quasi-isomorphic $\forall i \geq 0$

Definition

A GCDA (A, d_A) is said to be connected if $A^0 = \mathbb{Q}$

Theorem [Morgan 1978; Sullivan 1977]

Every connected differential graded algebra A has a minimal model, unique up to isomorphism.

- 1 Introduction
- 2 Graded Commutative Differential Algebras
 - Definition of a GCDA
 - The Sullivan model of a GCDA
 - An effective method to compute the minimal model
- 3 The Sullivan model of a space
 - From topology to algebra
 - Simplicial sets
 - The algebra $A_{PL}(K)$
 - The Sullivan model of $A_{PL}(K)$
 - The relation of the minimal model and the rational homotopy type of the space
- 4 My work
- 5 Applications

An effective method to compute the minimal model

The idea behind the method

The method is constructive, i.e., we build the model from lower to upper degrees, adding generators sequentially, in such a way the model satisfies the needed conditions.

A brief description of the algorithm (see [Manero & Marco (2020)])

Let (A, d_A) be the GCDA for which we want to get its minimal model, and let (M, d_M) be initially the trivial GCDA (with no generators).

- Let $k_0 > 0$ be the smallest degree for which $H^{k_0}(A)$ is not trivial.
- Take a basis $[a_0^{k_0}, \dots, a_{l_{k_0}}^{k_0}]$ of $H^{k_0}(A)$.

An effective method to compute the minimal model

A brief description of the algorithm (see [Manero & Marco (2020)])

- For each of these generators, $a_i^{k_0}$, add to M a generator of the same degree, $x_i^{k_0}$, and differential zero. Notice that for each addition, we want to remember the correspondence (i.e., we are building also the morphism $\varphi : M \rightarrow A$, by setting $\varphi(x_i^{k_0}) = a_i^{k_0}$).
- Now we know that φ induces an isomorphism in cohomology at degree k_0 .

An effective method to compute the minimal model

A brief description of the algorithm (see [Manero & Marco (2020)])

Next step is to check that φ induces a monomorphism in the cohomology group of degree $k_0 + 1$.

- if φ induces a monomorphism, at this point (M, d_M) is a k_0 -minimal model of (A, d_A) . We can repeat the same steps for degree $k_0 + 1$ to obtain a $(k_0 + 1)$ -minimal model of (A, d_A) .
- if φ does not induces a monomorphism in cohomology, we have to add to M generators of degree k_0 in order to kill those elements that should not live in cohomology.

Example

Example (2.1 from [Manero & Marco, 2020])

Let A be a GCDA generated by six generators of degree 1: $e_1 \dots e_6$ and one generator of degree 2: e_7 , with no relations and where the differential is given by:

$$d_A(e_1) = -e_1 \wedge e_6$$

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$$d_A(e_3) = -e_3 \wedge e_6$$

$$d_A(e_4) = -e_5 \wedge e_6$$

$$d_A(e_5) = d_A(e_6) = d_A(e_7) = 0$$

Example

Example (2.1 from [Manero & Marco, 2020])

We can compute the first cohomology groups

$$H^1(A) = \mathbb{Q} \langle e_5, e_6 \rangle$$

$$H^2(A) = \mathbb{Q} \langle e_4 \wedge e_5, e_4 \wedge e_6, e_7 \rangle$$

$$H^3(A) = \mathbb{Q} \langle e_4 \wedge e_5 \wedge e_6, e_5 \wedge e_7, e_6 \wedge e_7 \rangle$$

where $\mathbb{Q} \langle E_1, \dots, E_n \rangle$ denotes the vector subspace with basis $\{E_1, \dots, E_n\}$.

- 1 Introduction
- 2 Graded Commutative Differential Algebras
 - Definition of a GCDA
 - The Sullivan model of a GCDA
 - An effective method to compute the minimal model
- 3 The Sullivan model of a space
 - From topology to algebra
 - Simplicial sets
 - The algebra $A_{PL}(K)$
 - The Sullivan model of $A_{PL}(K)$
 - The relation of the minimal model and the rational homotopy type of the space
- 4 My work
- 5 Applications

The idea

Note

From now on, we assume that we are working with $\mathbb{K} = \mathbb{Q}$

Definition

Let X be a topological space. We denote by $C^*(X, \mathbb{Q})$ the cochain algebra of normalized singular cochains on X .

This algebra is almost never commutative.

Idea: Replace $C^*(X, \mathbb{Q})$ by a genuinely graded commutative cochain algebra, $A_{PL}(X, \mathbb{Q})$, and a cochain algebra quasi-isomorphism

$$A_{PL}(X, \mathbb{Q}) \xrightarrow{\cong} C^*(X, \mathbb{Q})$$

- 1 Introduction
- 2 Graded Commutative Differential Algebras
 - Definition of a GCDA
 - The Sullivan model of a GCDA
 - An effective method to compute the minimal model
- 3 The Sullivan model of a space
 - From topology to algebra
 - **Simplicial sets**
 - The algebra $A_{PL}(K)$
 - The Sullivan model of $A_{PL}(K)$
 - The relation of the minimal model and the rational homotopy type of the space
- 4 My work
- 5 Applications

Simplicial objects

Definition

A simplicial object K with values in a category $\mathfrak{C}$ is a sequence $\{K_n\}_{n \geq 0}$ of objects in $\mathfrak{C}$, together with $\mathfrak{C}$ -morphisms:

$$\delta_i : K_{n+1} \rightarrow K_n, \quad 0 \leq i \leq n+1, \quad s_j : K_n \rightarrow K_{n+1}, \quad 0 \leq j \leq n$$

that satisfy the identities:

$$\delta_i \delta_j = \delta_{j-1} \delta_i, \quad i < j$$

$$s_i s_j = s_{j+1} s_i, \quad i \leq j$$

$$\delta_i s_j = \begin{cases} s_{j-1} \delta_i, & i < j \\ id, & i = j, j+1 \\ s_j \delta_{i-1}, & i > j+1 \end{cases}$$

Simplicial objects

Definition

A simplicial set K is a simplicial object in the category of sets.

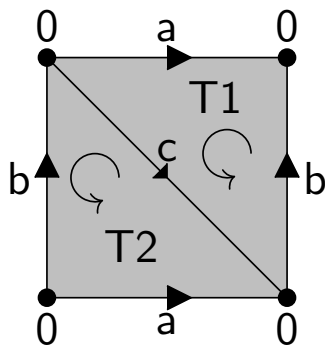
Definition

A simplicial cochain algebra A is a simplicial object in the category of cochain algebras.

Definition

A simplicial morphism $f : L \rightarrow K$ between two such simplicial objects is a sequence of $\mathfrak{C}$ -morphisms $\varphi_n : L_n \rightarrow K_n$ commuting with the δ_i and s_j .

Example of a simplicial set: $\mathbb{T}^2$



$$\{v_0\} \leftarrow \{a, b, c\} \leftarrow \{T_1, T_2\}$$

$$\delta_0(a) = \delta_0(b) = \delta_0(c) = v_0$$

$$\delta_1(a) = \delta_0 1(b) = \delta_1(c) = v_0$$

$$\delta_0(T_1) = \delta_1(T_2) = a$$

$$\delta_1(T_1) = \delta_2(T_2) = b$$

$$\delta_2(T_1) = \delta_0(T_2) = c$$

- 1 Introduction
- 2 Graded Commutative Differential Algebras
 - Definition of a GCDA
 - The Sullivan model of a GCDA
 - An effective method to compute the minimal model
- 3 The Sullivan model of a space
 - From topology to algebra
 - Simplicial sets
 - The algebra $A_{PL}(K)$
 - The Sullivan model of $A_{PL}(K)$
 - The relation of the minimal model and the rational homotopy type of the space
- 4 My work
- 5 Applications

The structure $A(K)$

Let K be a simplicial set, and let $A = \{A_n\}_{n \geq 0}$ be a simplicial cochain complex or a simplicial cochain algebra. Then

$$A(K) = \{A^p(K)\}_{p \geq 0}$$

is the cochain complex (or cochain algebra) defined as follows:

- $A^p(K)$ is the set of simplicial set morphisms from K to A^p .
Thus, an element $\Phi \in A^p(K)$ is a mapping that assigns to each n -simplex $\sigma \in K_n$ an element $\Phi_\sigma \in (A^p)_n$, such that

$$\Phi_{\delta_i(\sigma)} = \delta_i(\Phi_\sigma)$$

$$\Phi_{s_j(\sigma)} = s_j(\Phi_\sigma)$$

The structure $A(K)$

- Addition, scalar multiplication and the differential are given by

$$(\Phi + \Psi)_\sigma = \Phi_\sigma + \Psi_\sigma$$

$$(\lambda\Phi)_\sigma = \lambda\Phi_\sigma$$

$$(d\Phi)_\sigma = d(\Phi_\sigma)$$

- If A is a simplicial cochain algebra, multiplication in $A(K)$ is given by

$$(\Phi\Psi)_\sigma = \Phi_\sigma\Psi_\sigma$$

The simplicial commutative cochain algebra A_{PL}

Consider the free graded commutative algebra

$\wedge(t_0, \dots, t_n, y_0, \dots, y_n)$ in which the basis elements t_i have degree 0 and the basis elements y_j have degree 1. The differential is defined as:

$$d(t_i) = y_i$$

$$d(y_j) = 0$$

The simplicial commutative cochain algebra A_{PL}

Definition

We define the simplicial cochain algebra $A_{PL} = \{(A_{PL})_n\}_{n \geq 0}$ by:

- The cochain algebra $(A_{PL})_n$ is given by:

$$(A_{PL})_n = \frac{\wedge(t_0, \dots, t_n, y_0, \dots, y_n)}{\langle \sum t_i - 1, \sum y_j \rangle}$$

The simplicial commutative cochain algebra A_{PL}

Definition (cont.)

- The face and degeneracy morphisms are the unique cochain algebra morphisms:

$$\delta_i : (A_{PL})_{n+1} \rightarrow (A_{PL})_n, \quad 0 \leq i \leq n+1$$

$$s_j : (A_{PL})_n \rightarrow (A_{PL})_{n+1}, \quad 0 \leq j \leq n$$

satisfying

$$\delta_i = t_k \rightarrow \begin{cases} t_k, & k < i \\ 0, & k = i \\ t_{k-1}, & k > i \end{cases} \quad s_j = t_k \rightarrow \begin{cases} t_k, & k < j \\ t_k + t_{k+1}, & k = j \\ t_{k+1}, & k > j \end{cases}$$

The quasi-isomorphism $\oint$

If K is a simplicial set, then $A_{PL}(K)$ is a GCDA, where $C^*(K)$ is not, but the first one is constructed in order to be quasi-isomorphic to the latter:

Definition

The integration $\oint : A_{PL}(K) \rightarrow C^*(K)$ provides a natural quasi-isomorphism of cochain complexes.

$$\oint = \{\oint_n\}_{n \geq 0}$$
$$\oint_n : A_{PL}(K)_n^n \rightarrow C^n(K)$$

The Sullivan model of a space

Definition

We define the Sullivan model of a space X , as the minimal model of the graded commutative differential algebra $A_{PL}(X)$.

$$\begin{array}{ccccccc}
 & & M^{p-1} & \xrightarrow{d_M^{p-1}} & M^p & \xrightarrow{d_M^p} & M^{p+1} \\
 & \nwarrow \phi_{p-1} & & \nwarrow \phi_p & & \nwarrow \phi_{p+1} & \\
 & A_{pl}(K)^{p-1} & \xrightarrow{d_A^{p-1}} & A_{pl}(K)^p & \xrightarrow{d_A^p} & A_{pl}(K)^{p+1} & \\
 \nwarrow \phi_{p-1} & & & \nwarrow \phi_p & & \nwarrow \phi_{p+1} & \\
 C^{p-1}(K) & \xrightarrow{d_C^{p-1}} & C^p(K) & \xrightarrow{d_C^p} & C^{p+1}(K) & & \\
 & \downarrow & \downarrow & \downarrow & & & \\
 & & H^p(C^*(K)) & \xleftarrow{\phi_p^*} & H^p(A_{pl}(K)) & \xleftarrow{\phi_p^*} & H^p(M) \\
 & & \nwarrow \phi_p^* & & \nwarrow \phi_p^* & & \\
 & & H^p(C^*(K)) & \xleftarrow{\phi_p^* \circ \phi_p^*} & H^p(A_{pl}(K)) & \xleftarrow{\phi_p^* \circ \phi_p^*} & H^p(M)
 \end{array}$$

- 1 Introduction
- 2 Graded Commutative Differential Algebras
 - Definition of a GCDA
 - The Sullivan model of a GCDA
 - An effective method to compute the minimal model
- 3 The Sullivan model of a space
 - From topology to algebra
 - Simplicial sets
 - The algebra $A_{PL}(K)$
 - The Sullivan model of $A_{PL}(K)$
 - The relation of the minimal model and the rational homotopy type of the space
- 4 My work
- 5 Applications

My work

My work is focused on the development of a computer program that receives a simplicial set as input, and returns its minimal model.

- 1 Introduction
- 2 Graded Commutative Differential Algebras
 - Definition of a GCDA
 - The Sullivan model of a GCDA
 - An effective method to compute the minimal model
- 3 The Sullivan model of a space
 - From topology to algebra
 - Simplicial sets
 - The algebra $A_{PL}(K)$
 - The Sullivan model of $A_{PL}(K)$
 - The relation of the minimal model and the rational homotopy type of the space
- 4 My work
- 5 Applications

Applications

Work by Pablo Simón Isaza Peñaloza (2020)

CW-Decompositions of Plane Algebraic Curves and Milnor Fibers
of Non-Isolated Quasi-Ordinary Singularities

Thank you for your attention

-  Félix, Yves; Halperin, Stephen; Thomas, Jean-Claude. *Rational Homotopy Theory*. Springer Nature (2001)
-  Griffiths, P. and Morgan J. *Rational homotopy theory and differential forms*. Progress in Mathematics vol. 16, Birkhäuser (2013).
-  Manero V, Marco M.A. *Effective computation of degree bounded minimal models of GCDAs*. Journal of Software for Algebra and Geometry (2020)
-  Isaza Peñaloza, Pablo Simón *CW-Decompositions of Plane Algebraic Curves and Milnor Fibers of Non-Isolated Quasi-Ordinary Singularities*. Universidad Complutense de Madrid (2020)