Effective Computation of the Sullivan model of a topological space and its applications

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Structure of the talk

1- Definition of GCDA's

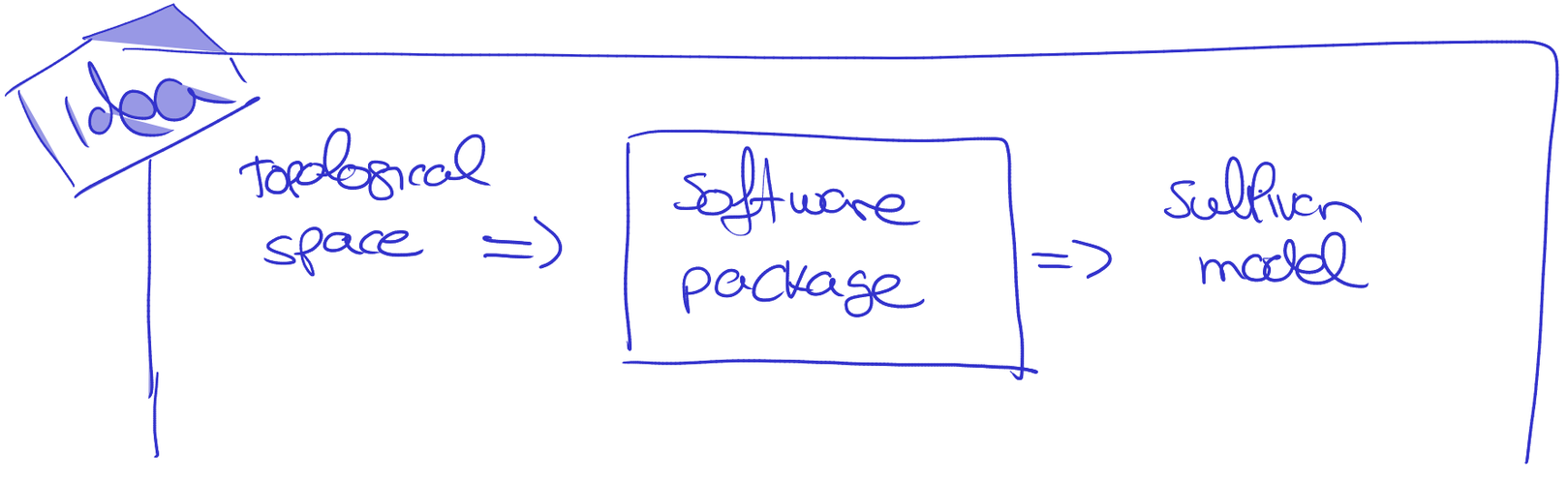
2- Definition of the minimal model of a GCDA

3- Computation of the model

4- Definition of $A_{PL}(K)$

5- Definition of the Sullivan model of the space

6- Properties of the model



Definition of a GCDA

A graded commutative differential algebra (or GCDA), A, over a ring $\mathcal{R}$, is a graded $\mathcal{R}$-algebra, $A = \bigoplus_{i=0}^{\infty} A^i$, together with an $\mathcal{R}$-linear map $d_A : A \to A$ that satisfies:

- $xy \in A^{i+j}$, $\forall x \in A^i$, $\forall y \in A^j$
- $xy = (-1)^{ij} yx$, $\forall x \in A^i$, $\forall y \in A^j$
- $d_A^2 = 0$
- $d_A(x) \in A^{i+1}$, $\forall x \in A^i$
- $d_A(xy) = d_A(x) \cdot y + (-1)^i x d_A(y)$, $\forall x \in A^i$, $\forall y \in A$

From now on, every algebra is a GCDA

(Relative) Sullivan algebras

Def: A relative Sullivan algebra is a commutative cochain algebra

$$(B \otimes \Lambda V, d)$$

with the following properties:

- the "base algebra" $(B, d) = (B \otimes 1, d)$ is a subcochain algebra and connected, i.e., $H^0(B, d) = \mathbb{Q}$
- the algebra ΛV is connected and built upon a graded vector space $1 \otimes V = V = \{V^p\}_{p \geq 1}$

(Relative) Sullivan algebras

- there exists a filtration

$$V = \bigcup_{k=0}^{\infty} V(k)$$

of graded subspaces

$$V(0) \subseteq V(1) \subseteq V(2) \subseteq \cdots \subseteq V$$

such that

$$d: V(0) \longrightarrow B$$

and

$$d: V(k) \longrightarrow B \otimes \Lambda(V(k-1)), \text{ for } k \geq 1$$

(Relative) Sullivan algebras

A Sullivan algebra is a relative Sullivan algebra with $(B, d) = (\mathbb{Q}, 0)$

A relative Sullivan algebra is minimal if

$$\operatorname{im} d \subseteq B^{\geq 1} \otimes \Lambda V + B \otimes \Lambda^{\geq 2} V$$

For a Sullivan algebra, this is equivalent to

$$\operatorname{im} d \subseteq \Lambda^{\geq 2} V$$

(Relative) Sullivan algebras: Examples

1) A CDGA that is not a Sullivan algebra: [Manero & Marco 2020]

Let A be a CDGA generated by $\langle e_1, \dots, e_7 \rangle$

where $|e_i| = 1$, for $i \leq 6$

$|e_7| = 2$

and the differential is given by:

$$d(e_1) = -e_1 \wedge e_6$$
$$d(e_2) = -e_2 \wedge e_6$$
$$d(e_3) = -e_3 \wedge e_6$$
$$d(e_4) = -e_5 \wedge e_6$$
$$d(e_5) = d(e_6) = d(e_7) = 0$$

It is not a Sullivan algebra because the differentials of e_1, e_2 and e_3 are given as a product of themselves.

Sullivan algebras: Examples

2) A Sullivan algebra (not relative, not minimal)

$$(\Lambda(y,t), d) \quad \begin{matrix} |y| = 3 & dy = 0 \\ |t| = 2 & dt = y \end{matrix}$$

It is a Sullivan algebra but it is not minimal because it is not satisfied the condition $\operatorname{Im} d \subseteq \Lambda^{\geq 2} V$

3) A minimal Sullivan algebra

$$(\Lambda(x,y,z), d) \quad |x| = |y| = |z| = 1$$

$$dx = dy = 0$$

$$dz = xy$$

Definition of the minimal model of a GCDA

Let (A, d_A) be a connected ($H^0(A) = \mathbb{Q}$) GCDA.

Def: A <u>minimal model</u> of (A, d_A) is a minimal Sullivan algebra (M, d_M), together with a <u>quasi-isomorphism</u> φ (induces $\cong$ in cohomology)

$$\varphi : M \longrightarrow A$$

$$\varphi^* : H^*(M) \xrightarrow{\cong} H^*(A)$$

<u>thm</u>: Every connected GCDA, A, has a minimal model (M_A, φ_A)

<u>thm</u>: the minimal model of a GCDA is unique up to isomorphism

The algorithm for the minimal model

Let (A, d_A) be a connected ($H^0(A, d_A) = \mathbb{Q}$) commutative cochain algebra

$$\varphi: (M, d_M) \longrightarrow (A, d_A)$$

Step 1) Find the first $i \geq 1$ for which $H^i(A, d_A) \neq 0$

Add generators x_j to M^i ($|x_j| = i$) in such a way that $\varphi^i_*: H^i(M, d_M) \longrightarrow H^i(A, d_A)$ is an isomorphism.

The algorithm for the minimal model

Step 2)

If $\varphi_*^{i+1} : H^{i+1}(M, d_M) \longrightarrow H^{i+1}(A, d_A)$ is injective

$\Rightarrow$ go back to step 1 and repeat the process for degree $i+1$

else (φ_*^{i+1} not injective)

Add to M^i generators y_j in such a way you make φ_*^{i+1} injective.

Example: the minimal model of $(H^*(S^n;\mathbb{Q}), d=0)$ for $n \geq 2$

$$H^q(S^n;\mathbb{Q}) = \begin{cases} \langle h^q \rangle & \text{if } q = \{0, n\} \\ 0 & \text{otherwise} \end{cases} \qquad \text{with } (h^n)^2 = 0$$

First, consider 'n' odd:

$$h^n \xleftarrow{\varphi} x_0^n \xrightarrow{d_M} 0$$

as $|x_0^n|$ is odd, $(x_0^n)^2 = 0$ so we have that $\boxed{M \simeq H^*(S^n)}$

Example: the minimal model of $H^*(S^n)$

Consider now the case 'n' even

$$h^n \xleftarrow{\varphi} x_0^n \xrightarrow{d_M} 0$$

Now we have an isomorphism at degree 'n'

but because 'n' is even, we have

$$H^{2n}(M) = \mathbb{Q}\langle [(x_0^n)^2] \rangle$$

but $$H^{2n}(S^n) = 0$$

So we need to add a generator of degree 2n-1 in order to make this cohomology trivial

$$0 \xleftarrow{\varphi} y_0^{2n-1} \xrightarrow{d_M} (x_0^n)^2$$

So $$\boxed{M = \mathbb{Q}\langle x_0^n, y_0^{2n-1} \rangle}.$$

The Sullivan model of a space

We assume that our topological space can be expressed as a simplicial complex / set.

The idea: we are going to construct an algebra

For that reason, we need

1) A topological space expressed as a simplicial set

2) the algebra of polynomial differential forms

3) A way to relate both structures

The Sullivan model of a space

Def: A simplicial object K with values in a category $\mathcal{C}$ is a sequence $\{K_n\}_{n\geq 0}$ of objects in $\mathcal{C}$, together with $\mathcal{C}$-morphisms:

$$\boxed{\partial_i : K_{n+1} \longrightarrow K_n, \quad 0\leq i\leq n+1} \quad , \quad \boxed{s_j : K_n \longrightarrow K_{n+1}, \quad 0\leq j\leq n}$$

that satisfy the identities:

$$\partial_i\partial_j = \partial_{j-1}\partial_i, \quad i<j$$

$$s_i s_j = s_{j+1} s_i, \quad i\leq j$$

$$\partial_i s_j = \begin{cases} s_{j-1}\partial_i, & i<j \\ id, & i=j,j+1 \\ s_j\partial_{i-1}, & i>j+1 \end{cases}$$

Examples

- Simplicial sets
- Simplicial cochain algebras

the Sullivan model of a space

Def: A simplicial morphism $\varphi : L \to K$ between two such simplicial objects is a sequence of $\mathcal{C}$-morphisms φ_n, commuting with the faces and degeneracies.

$$\begin{array}{ccccc} L_{n-1} & \underset{s_j}{\overset{\partial_i}{\rightleftarrows}} & L_n & \underset{s_j}{\overset{\partial_i}{\rightleftarrows}} & L_{n+1} \\ \downarrow{\scriptstyle \varphi_{n-1}} & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_{n+1}} \\ K_{n-1} & \underset{s_j}{\overset{\partial_i}{\rightleftarrows}} & K_n & \underset{s_j}{\overset{\partial_i}{\rightleftarrows}} & K_{n+1} \end{array}$$

the structure $A(K)$

Let K be a simplicial set, and let $A = \{A_n\}_{n\geq 0}$ be a simplicial cochain complex or a simplicial cochain algebra.

then,

$$A(K) = \{A^p(K)\}_{p\geq 0}$$

is the cochain complex (or cochain algebra) defined as follows:

$A = \{A_n\}_{n\geq 0}$

$K = \{K_n\}_{n\geq 0}$

$$\cdots \leftarrow K_{n-1} \leftarrow K_n \leftarrow K_{n+1} \leftarrow \cdots$$

$$\begin{array}{ccccc}
A_n^{p-1} & \rightleftarrows & A_{n+1}^{p-1} & \rightleftarrows & A_{n+2}^{p-1} \\
\downarrow d & & \downarrow d & & \downarrow d \\
A_n^{p} & \rightleftarrows & A_{n+1}^{p} & \rightleftarrows & A_{n+2}^{p} \\
\downarrow d & & \downarrow d & & \downarrow d \\
A_n^{p+1} & \rightleftarrows & A_{n+1}^{p+1} & \rightleftarrows & A_{n+2}^{p+1}
\end{array}$$

the structure $A^P(K)$

- $A^P(K)$ is the set of simplicial set morphisms from K to A^P.

thus, an element $\Phi \in A^P(K)$ is a mapping that assigns to each n-simplex $\sigma \in K_n$, an element $\Phi_\sigma \in (A^P)_n$ such that

$$\Phi(\partial_i(\sigma)) = \partial_i(\Phi(\sigma))$$

$$\Phi(s_j(\sigma)) = s_j(\Phi(\sigma))$$

the structure $A(K)$

- Addition, scalar multiplication and the differential are given by:

$$(\Phi + \Psi)(\sigma) = \Phi(\sigma) + \Psi(\sigma)$$

$$(\lambda\Phi)(\sigma) = \lambda\Phi(\sigma)$$

$$(d\Phi)(\sigma) = d(\Phi(\sigma))$$

- If A is a simplicial cochain algebra, multiplication in $A(K)$ is given by

$$(\Phi \cdot \Psi)(\sigma) = \Phi(\sigma) \cdot \Psi(\sigma)$$

The simplicial commutative cochain algebra A_{PL}

Consider the free graded commutative algebra

$$\Lambda(t_0, \ldots, t_n, y_0, \ldots, y_n)$$

$$\deg(t_i) = 0$$

$$\deg(y_i) = 1$$

the differential $= \begin{cases} d(t_i) = y_i \\ d(y_i) = 0 \end{cases}$

the simplicial commutative cochain algebra A_{PL}

the simplicial cochain algebra

$$A_{PL} = \{(A_{PL})_n\}_{n \geq 0}$$

is defined as:

- the cochain algebra $(A_{PL})_n$ is given by:

$$(A_{PL})_n = \frac{\wedge(t_0, \ldots, t_n, y_0, \ldots, y_n)}{\langle \sum t_i - 1, \sum y_i \rangle}$$

- Faces and degeneracies:

$$\partial_i = t_k \longrightarrow \begin{cases} t_k, & k < i \\ 0, & k = i \\ t_{k-1}, & k > i \end{cases} \qquad s_j = t_k \longrightarrow \begin{cases} t_k, & k < j \\ t_k + t_{k+1}, & k = j \\ t_{k+1}, & k > j \end{cases}$$

the Sullivan model of a space

Given a topological space X as a simplicial set, we define the Sullivan model of X, as the minimal model of $A_{PL}(X)$

When X is simply-connected, the minimal model give us information about the rational homotopy type of the space

$$\#M^p = \dim(\Pi_p(X) \otimes \mathbb{Q})$$

Applications

1) Isaza, Artal, Gonzalez Perez and Carmona

CW-Decomposition of Plane Algebraic Curves and Milnor Fibers of Non-Isolated Quasi-Ordinary Singularities

2) Study of formality

Formal space

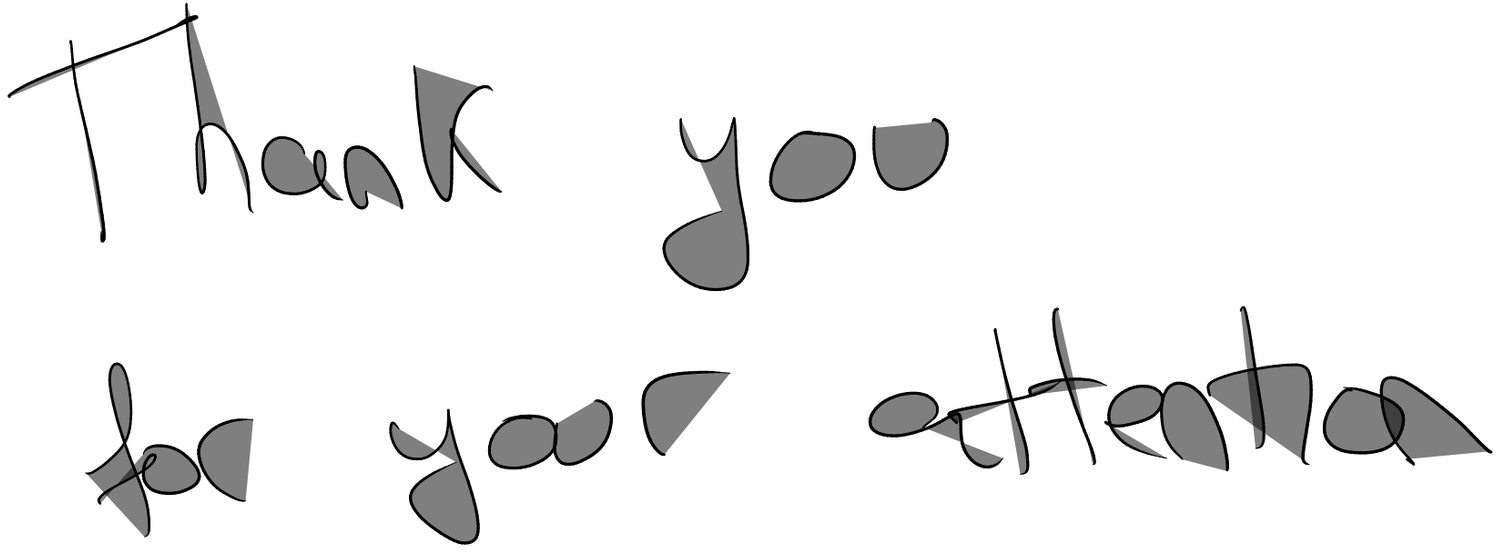
M_X

$A_{PL}(X)$ $\quad$ $H^*(X)$

Non-formal space

$M_X \not\simeq M_{H(X)}$

$A_{PL}(X)$ $\quad$ $H^*(X)$


Thank you
for your attenta

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