ON GERMS OF HYPERSURFACES WITH MULTIPLICITY 3

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Consider a the germ of a multiplicity 3 singular point of hypersurface. An equation for such a germ it is given by a monic Weierstrass polynomial $F = z^3 + a_1 z^2 + a_2 z + a_3 \in \mathbb{C}[[\mathbf{x}]][z]$ of degree 3 with respect to z. Assume that $D_z F$ are not zero, and that F is degenerate with respect to its Newton polyhedron. If it is non degenerate, the partial resolution associated with the dual of the Newton polyhedron of F gives plenty information of the geometry and topology of the singular point of the hypersurface defined by F.

We propose the following general method.

First step: By means of a Tschirnhaus transformation, reduce F to a monic Weierstrass equation $F = z^3 - 3b_2 z + 2b_3$ that does not have terms of degree 2 with respect to z. Assume that the coefficients of degree one and zero with respect to z are both non zero, i.e., the product $b_2 b_3 \neq 0$. If either $b_2 = 0$ and $b_3 \neq 0$ or $b_2 \neq 0$ and $b_3 = 0$ it reduces to the ase of a suspension that we have studied elsewhere. See [**?**].

Second step: Consider an embedded resolution $\pi : X \to \mathbb{C}^2$ of the reduced divisor given by the product $G = b_2 b_3 D_z F$. Consider the usual stratification of the pair $(X, (\pi \circ G)^{-1}\{G = 0\})$ defined in terms of decomposition $\cup_{j\in J} E_j$ of $(\pi \circ G)^{-1}\{G = 0\}$ into irreducible components. Pulling back F under the map $\pi \times id$, we obtain for each stratum E_I° a *quasi-trinomial local expression* for F. Section 3 clasifies the 5 posible quasi-trinomial local forms having terms of degrees 3, 1, and 0 with respect to z. These are the quasi-trinomials forms that appear at this step because we have previously made a Tschirnhaus transformation. Notice that the local expression is quasi-trinomial due to the fact that π resolves both b_2 and b_3. Moreover, the quasi-trinomial local form is quasi-ordinary because π also resolves the discriminant $D_z F$. The local quasi-trinomial expression will be non degenerate with respect to its Newton polyhedron unless the monomials M_1, M_2 and their coefficients c_1 and c_2 satisfy a precise condition. In principle, this precise condition can be defined on a subset of a stratum of E_I°, with $\emptyset \neq I \subset J$. If it is the case, we consider the corresponding refinement of the stratification. Moreover, the degeneracies of the degenerate trinomial local case are either on horizontal faces or of the form $(z - af(x, y))(z - bf(x, y))$ with $a, b \in \mathbb{C}^*$, and $f \in \mathbb{C}[x, y]$.

Third step: Make translations of the type $z \mapsto z + h(x, y)$, where $h(x, y)$ is determined by the local quasi-ordinary expansion, to obtain a new set of coordinates such that the local form is non degenerate. Notice that after the translation terms of degree 2 respect to z appear. In fact, we obtain a new type of quasi-trinomial local form with terms of degree 3, 2, and 0 with respect to z.

Fourth step: Use the stratification principle with the stratification introduced in the Second Step, the quasi-trinomial local forms and, if needed, the corresponding local differential forms to compute *additive / multiplicative local invariants* of the singular point defined by f at o. For instance, we will interested in the A'Campo zeta function and in the topological zeta function.

A couple of remarks are in order after these four steps.

First, these steps achieve a resolution of the meromorphic function defined by $\frac{b_2^3}{b_3^2}$ and therefore the irreducible components E_j of $(\pi \circ F)^{-1}\{F = 0\}$ can be classified into 5 different types according to the image of e_j under the meromorphic function $\mathcal{M}$ defined by $\frac{b_2^3}{b_3^2}$. There are components E_j that are mapped to 0 called zeros, components E_j that are mapped to ∞ called poles, components E_j that are mapped surjectively onto $\mathbb{P}^1$ called dicritical, components that are mapped to the value 1, and finally components that are mapped to other values different than 0, 1, and ∞. The image of a component E_j is determined by the orders of vanishing of f, g and $\mathcal{M} = \frac{b_2^3}{b_3^2}$ along the divisor E_j and (in some cases by) their pattern of intersection.

Second, the trinomial part of any quasi-trinomial local form, given by the formal restriction of the quasi-trinomial form to its Newton polyhedron, is determined by the valuations associated with f, g and $\mathcal{M} = \frac{b_2^3}{b_3^2}$ too.

The following reductions are possible to perform the Fourth Step.

Reduction A: It is enough to consider the associated trinomial local forms instead of the quasi-trinomial local forms.

Reduction B: The strata produced by the strict transforms of b_2 and b_3 can be omitted, i.e., the contribution to the invariant due to a 0-dimensional stratum determined by an irreducible component of the strict transform of b_2 (resp. b_3) equals the contribution of a single point of the 1-dimensional stratum to whose adherence the 0-dimensional stratum belongs.

Reduction C: Bamboos consisting of irreducible components that are mapped to 0 (resp. to ∞ and containing a (-1)-curve can be omitted too.

The upshot of Reductions A, B, and C is that it is enough to consider the stratification associated to the minimal resolution $\pi' : X' \to \mathbb{C}^2$ of D_zF together with the trinomial local forms, and if needed the corresponding differential forms to compute the invariants of the singular point defined by f at 0.

Reduction D: It is enough to consider a minimal $\mathbb{Q}$-resolution of the discriminant D_ZF.

After all these reductions, we will prove that ...

1. Power series of multiplicity three

Consider a the germ of a multiplicity 3 singular point of hypersurface. An equation for such a germ it is given by a monic Weierstrass polynomial $F = z^3 + a_1z^2 + a_2z + a_3 \in \mathbb{C}[[\mathbf{x}]][z]$ of degree 3 with respect to z. Assume that D_zF is not zero, and that F is degenerate with respect to its Newton polyhedron.

Definition 1.1. The Tschirnhaus transformation associated to F is the transformation given by $z \mapsto z - \frac{a_1}{3}$.

After the Tschirnhaus transformation, the monic Weierstrass polynomial $F(x, y, z - a_1/3)$ can be written as

$$z^3 - 3b_2z + 2b_3$$

with

$$b_2 = \frac{a_1^2}{9} - \frac{a_2}{3} \quad \text{and} \quad b_3 = \frac{a_1^3}{27} - \frac{a_1 a_2}{6} + \frac{a_3}{2}$$

and its the discriminant $D_z F$ of F with respect to F is of the form $108(b_2^3 - b_3^2)$.

Denote the decompositions of b_2 and b_3 into irreducible components as

$$b_2 = \prod_{i=1}^{n} h_i^{n_i} \prod_{j=1}^{m} k_j f \qquad \text{and} \qquad b_3 = \prod_{i=1}^{n} h_i \prod_{j=1}^{m} k_j^{m_j} g \tag{1.0.1}$$

where h_i, k_j are irreducible and h_i, k_j, f, g are pairwise coprime and $n_i, m_j \geq 1$.

Proposition 1.2. *The discriminant $D_z F$ is of the form*

$$\prod_{i=1}^{n} h_i^2 \prod_{j=1}^{m} k_j^3 \left(\prod_{i=1}^{n} h_i^{3n_i - 2} f^3 - \prod_{j=1}^{m} h_i^{2m_j - 3} g^2 \right).$$

Proposition 1.3. *The singular point is isolated if and only if the disciminant $D_z F$ is reduced.*

Proposition 1.4. *The monic Weierstrass polynomial $z^3 - 3b_2 z + 2b_3$ is non degenerate unless $M_1(c_1 + A)$ and $b_3 = M_2(c_2 + B)$ with $M_1^3 = M_2^2$ and $2c_1 = 3c_2$*

2. Jung´s method for power series of multiplicity three

Consider an embedded resolution $\pi : X \to \mathbb{C}^2$ of the reduced divisor given by the product $G = b_2 b_3 D_z F$. Consider the usual stratification of the pair $(X, (\pi \circ G)^{-1}\{G = 0\})$ defined in terms of decomposition $\cup_{j \in J} E_j$ of $(\pi \circ G)^{-1}\{G = 0\}$ into irreducible components, i.e., for each subset $I \subset J$ we define

$$E_I^\circ = \bigcap_{i \in I} E_j \setminus \bigcup_{j \notin I} E_j.$$

Definition 2.1. For each irreducible component E_j of of $(\pi \circ G)^{-1}\{G = 0\}$ we define the numbers

$$N_j^0 = \mathrm{ord}_{E_j} \pi^* b_2, \quad N_j^\infty = \mathrm{ord}_{E_j} \pi^* b_3, \quad N_j^\Delta = \mathrm{ord}_{E_j} \pi^* D_z F, \quad \text{and} \quad \nu_j = \mathrm{ord}_{E_j} \pi^* \omega - 1$$

where $\omega = dx \wedge dy$ is the standard volume form of $\mathbb{C}^2$.

Proposition 2.2. *Let $\pi' : X' \to \mathbb{C}^2$ be the minimal resolution of $D_z F$. Then, π' is a resolution of the meromorphic function $\mathcal{M} : \mathbb{C}^2 \to \mathbb{P}^1$ defined by $\frac{b_2^3}{b_3^2}$ too.*

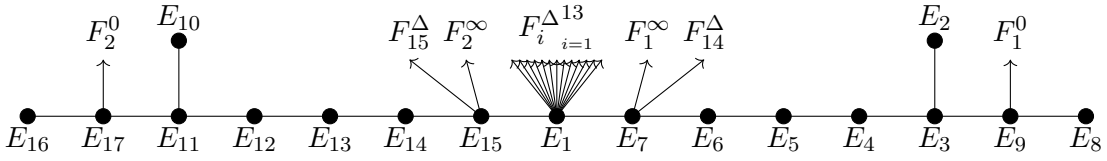
There are four possibilities for the behaviour of the numbers N_j^0, N_j^∞ and N_j^Δ associated with an irreducible component E_j of $(\pi \circ G)^{-1}\{G = 0\}$

- $3N_j^0 > 2N_j^\infty = N_j^\Delta$ and it is called a *zero* because in this case $\mathcal{M}(E_j) = 0$;
- $2N_j^\infty > 3N_j^0 = N_j^\Delta$ and it is called a *pole* because in this case $\mathcal{M}(E_j) = \infty$;
- $3N_j^0 = 2N_j^\infty = N_j^\Delta$ and in this case either it is called *dicritical* because $\mathcal{M}(E_i) = \mathbb{P}^1$ or it is mapped constantly to a value different than 0, 1 or ∞ under $\mathcal{M}$;
- $3N_j^0 = 2N_j^\infty < N_j^\Delta$ and in this case it is mapped constantly to 1 under $\mathcal{M}$.

In order to distinguish between dicritical divisors and divisors that are mapped constantly to a value different than 0, 1 or ∞ one notices that connected components of dicritical divisors must intersect both to zeros and poles and connected components of divisors mapped constantly to a value different than 0, 1 or ∞ do not intersect zeros or poles but do intersect dicritical divisors.

Definition 2.3. For a divisor of the last type, the difference $N_j^\Delta - 3N_j^0 = N_j^\Delta - 2N_j^\infty$ is called the *excess of the discriminant* and it is denoted by M_j^Δ.

Example 2.4. Consider $F = z^3 - 3b_2z + 2b_3$ with $b_2 = ((y^2+x^3)^2+x^5y)((x^2+y^3)^2+y^5x)$ and $b_3 = (y^3+x^4)(x^3+y^4)$. The minimal resolution π of the product $G = b_2b_3D_zF$ is obtained after 15 blow-ups. The following figure and table give the dual resolution graph of G and its associated numerical data.



	E_1	E_2, E_{10}	E_3, E_{11}	E_4, E_{12}	E_5, E_{13}	E_6, E_{14}	E_7, E_{15}	E_8, E_{16}	E_9, E_{17}
N_i^f	**8**	10	20	28	36	44	**52**	21	42
N_i^g	**12**	**13**	**26**	**39**	**52**	**65**	**78**	**26**	**52**
N_i^Δ	25	**26**	**52**	**78**	**104**	**130**	**156**	**52**	**104**
ν_i	2	3	5	7	9	11	13	6	11
$\chi()$	-13	1	-1	2	2	2	-4	1	-1
E_i^2	-7	-2	-2	-2	-2	-2	-1	-2	-1

The divisor E_1 is mapped to the value 1, the divisors E_7 and E_{13} are dicriticals, and the rest of exceptionals divisors are poles. The 13 arrows F_i^Δ with $i = 1, \ldots, 13$ attached to the divisor E_1 correspond to irreducible components of the strict transforms of D_zF, the arrows attachef to the divisor E_7 (resp. E_{13}) correspond to an irreducible component strict transform of b_3 and another irreducible component of D_zF.

$$\pi^{-1}\{\mathcal{M}=0\} = 2(\sum_{1,2} F_i^0 + \sum_{6,14} E_i) + 4\sum_{2,5,10,13} E_i + 6\sum_{4,12} E_i + 8\sum_{3,11} E_i + 11\sum_{8,16} E_i + 22\sum_{9,17} E_i,$$

$$\pi^{-1}\{\mathcal{M}=\infty\} = 3F_1^\infty + 3F_2^\infty, \quad \pi(E_7) = \pi(E_{15}) = \mathbb{P}^1, \quad \pi^{-1}\{\mathcal{M}=1\} = \sum_{i=1}^{15} F_i^\Delta + E_1$$

The stratification associated with the resolution π consist of 35 strata $E_{i,j}^\circ$ of dimension 0 and 8 strata E_i° of dimension 1 and $\chi(E_i^\circ) \neq 0$. The former correspond to the 16 pairs (i,j) listed belllow

$$\begin{array}{cccccccc} (1,7) & (2,3) & (3,4) & (3,9) & (4,5) & (5,6) & (6,7) & (8,9) \\ (1,15) & (10,11) & (11,12) & (11,17) & (12,13) & (13,14) & (14,15) & (16,17) \end{array}$$

the 13 intersections $E_1 \cap F_i^\Delta$ for $i = 1, \ldots, 13$ and the 6 intersections

$$E_7 \cap F_1^\infty, \quad E_7 \cap F_{14}^\Delta, \quad E_9 \cap F_1^0, \quad E_{15} \cap F_2^\infty, \quad E_{15} \cap F_{14}^\Delta, \quad E_{17} \cap F_2^0$$

and the latter correspond with the indexes

$$1, \quad 2, \quad 3, \quad 7, \quad 8, \quad 10, \quad 11, \quad 15.$$

Notice that the minimal resolution π' of the discriminant D_zF is obtained after just 11 blow-ups. The blowups corresponding to divisors E_8, E_9, E_{14} and E_{15} are needed to resolve b_2.

Let us consider the birational map $\tilde{\pi} := (\pi, id) : \tilde{X} := X \times \mathbb{C} \to \mathbb{C}^3$, where $id : \mathbb{C} \to \mathbb{C}$ is the identity map, and the pullback of F under $\pi \times id$. The pullback π^*F can be expressed in terms of the pullbacks of b_2 and b_3 as

$$z^3 - 3\pi^*b_2 z + 2\pi^*b_3. \tag{2.0.1}$$

Over each non empty strata E_I° of X created by the resolution π, for the pullback of F under the birational map $\pi \times id$. Since π is a resolution of both b_2 and b_3, over each non empty strata E_I° of X created by the resolution π, the terms $(\pi^*b_3)_{|_I}$ and $(\pi^*b_3)_{|_I}$ can be reexpressed in terms of the multiplicities N_i^f and N_i^g with $i \in I$ as

$$(\pi^*b_2)_{|_I} = x_1^{N_{i_1}^0} y_1^{N_{i_2}^0} u_1(x_1, y_1) \quad \text{and} \quad (\pi^*b_3)_{|_I} = x_1^{N_{i_1}^\infty} y_1^{N_{i_2}^\infty} u_1(x_1, y_1) \tag{2.0.2}$$

where u_1, u_2 are units in $\mathbb{C}\{x_1, y_1\}$. Hence, we have expressions

$$z^3 - 3x_1^{N_{i_1}^0} y_1^{N_{i_2}^0} u_1(x_1, y_1) z + 2x_1^{N_{i_1}^\infty} y_1^{N_{i_2}^\infty} u_2(x_1, y_1), \tag{2.0.3}$$

for each non empty strata E_I°. Notice that these expressions are quasi-ordinary because π is also a resolution of D_zF and therefore $D_z(\pi^*F)_{|_I}$ is the product of a monomial in the variables x_1 and y_1 and a unit in $\mathbb{C}\{x_1, y_1\}$.

According to Section 3 the expression (2.0.3) is non degenerate with respect to its Newton polyhedron except for the case described in Proposition 1.4, i.e., the case when the there is a divisor E_i with $i \in I$ such that $\mathcal{M}(E_i) = 1$ or equivalently $3N_i^0 = 2N_i^\infty < N_i^\Delta$.

Remark 2.5. If the strata $E_{i,j}^\circ \neq \emptyset$ and $\mathcal{M}(E_i) = 1$ then either $\mathcal{M}(E_j) = 1$ or E_j is dicritical, i.e., $\mathcal{M}(E_j) = \mathbb{P}^1$.

Proposition 2.6. *There exist a change of coordinates of the form*

$$x_1 = x_2, \quad y_1 = y_2, \quad z = z_1 - x_2^{\frac{N_i^0}{2}} y_2^{\frac{N_j^0}{2}} + \cdots$$

such that (2.0.3) becomes either

$$z_1^3 + x_2^{\frac{N_i^0}{2}} y_2^{\frac{N_j^0}{2}} u_1'(x_2, y_2) z_1^2 + x_2^{N_i^\infty + M_i^\Delta} y_2^{N_i^\infty + M_i^\Delta} u_2'(x_2, y_2)$$

if both $\mathcal{M}(E_i) = \mathcal{M}(E_j) = 1$ *or*

$$z_1^3 + x_2^{\frac{N_i^0}{2}} y_2^{\frac{N_j^0}{2}} u_1'(x_2, y_2) z_1^2 + x_2^{N_i^\infty + M_i^\Delta} y_2^{N_j^\infty} u_2'(x_2, y_2)$$

if $\mathcal{M}(E_i) = 1$ *and* E_j *is dicritical*

Definition 2.7. The trinomial form f_I associated with the stratum E_I° is

$$\begin{cases} z^3 - x^{N_i^\infty}, & \mathcal{M}(E_i) \neq 1, \infty \\ z^3 - x^{N_i^0} z + x^{N_i^\infty}, & \mathcal{M}(E_i) = \infty \\ z^3 - x_2^{\frac{N_i^0}{2}} + x^{N_i^\infty + M_i^\Delta}, & \mathcal{M}(E_i) = 1, \end{cases}$$

in the case $I = \{i\}$ and

$$\begin{cases} z^3 + x^{N_i^\infty} y^{N_j^\infty}, & \mathcal{M}(E_i), \mathcal{M}(E_j) \neq 1, \infty \\ z^3 + x^{\frac{N_i^0}{2}} y^{\frac{N_j^0}{2}} z^2 + x^{N_i^\infty + M_i^\Delta} y^{N_j^\infty + M_i j^\Delta}, & \mathcal{M}(E_i) = \mathcal{M}(E_j) = 1 \\ z^3 + x^{\frac{N_i^0}{2}} y^{\frac{N_j^0}{2}} z^2 + x^{N_i^\infty + M_i^\Delta} y^{N_j^\infty}, & \mathcal{M}(E_i) = 1, \mathcal{M}(E_j) = \mathbb{P}^1 \\ z^3 + x^{N_i^0} y^{N_j^0} z + x^{N_i^\infty} y^{N_j^\infty}, & \text{else} \end{cases}$$

in the case $I = \{i, j\}$.

3. The quasi-trinomial case

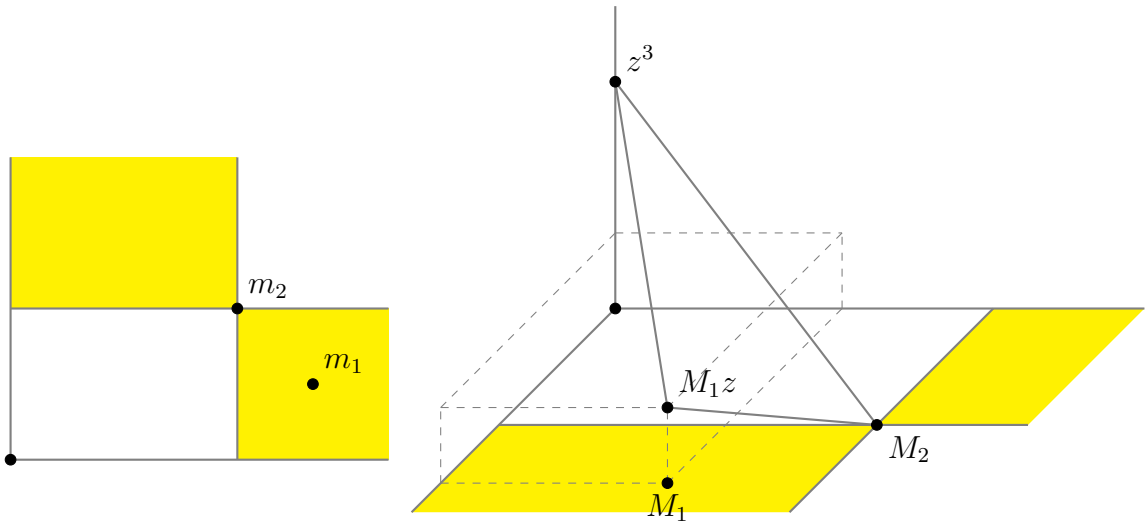
In this section we consider the case of

$$F = z^3 + M_1(c_1 + A)z + M_2(c_2 + B) \in \mathbb{C}[[\mathbf{x}]][z]$$

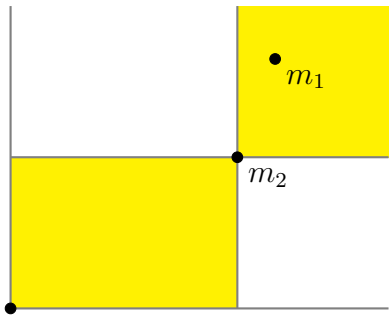
where $M_1 = \mathbf{x}^{m_1}$ and $M_2 = \mathbf{x}^{m_2}$ are monomials, $c_1, c_2 \in \mathbb{C}^*$ and $A, B \in (\mathbf{x})\mathbb{C}[[\mathbf{x}]]$

We distinguish the following two cases

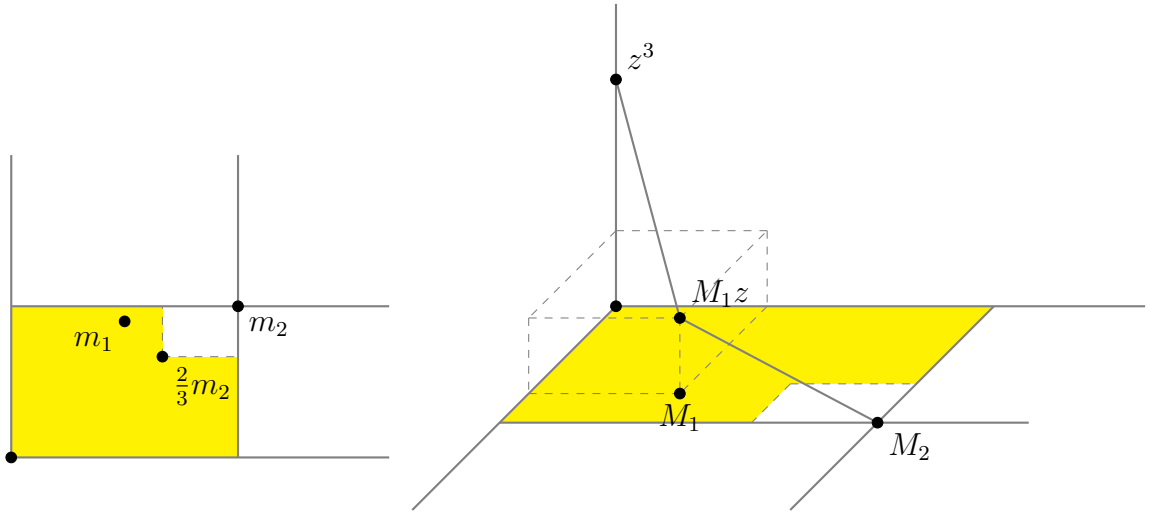
- Assume that the monomials M_1 and M_2 are not related by the divisibility relation, i.e., neither $M_1 \nmid M_2$ nor $M_2 \nmid M_1$. In this case, the Newton polyhedra of F contains a unique two dimensional compact face determined by the monomials z^3, $M_1 z$ and M_2. Moreover, F is non degenerate with respect to its Newton polyhedra.



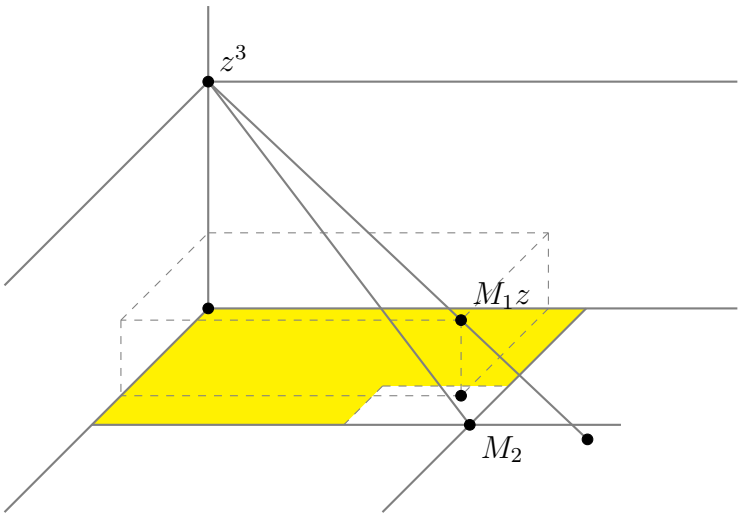
- Assume that the monomials M_1 and M_2 are related by the divisibility relation, i.e., either $M_1 | M_2$ or $M_1 | M_2$. In this case the Newton polyhedron of F is a polygonal path. We distinguish the following cases.



(1) If the point $3m_1$ does not belong to the set $2m_2 + (\mathbb{R}_{\geq 0})^2$, the Newton polyhedron of F consists of two segments. The first one is determined by the monomials z^3 and $M_1 z$ and the second is determined by the monomials $M_1 z$ and M_2. it follows that F is non degenerate with respect to its newton polyhedron.



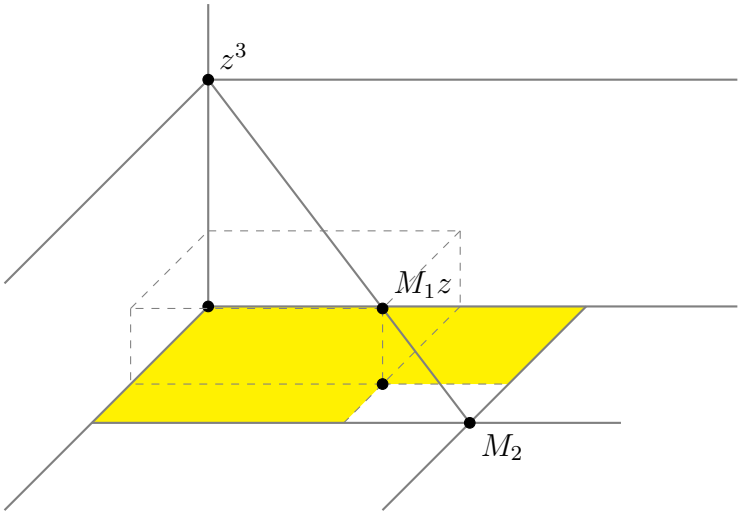
(2) If the point $3m_1$ belongs to the set $2m_2 + (\mathbb{R}_{\geq 0})^2$, the Newton polyhedron of F consists of one segment with vertex corresponding to the monomials z^3 and M_2. We distinguish to further cases.
   (a) If $M_1^3 \neq M_2^2$, the point corresponding to the monomial $M_1 z$ is an interior point of the Newton polyhedron of F. It follows that F is non degenerate with respect to its Newton polyhedron.



(b) If $M_1^3 = M_2^2$, the point corresponding to the monomial $M_1 z$ lies in the unique dimensional compact face of the Newton polyhedron of F. The face equation is equal to

$$z^3 + c_1 z + c_2.$$

Therefore, F is non degenerate with respect to its Newton polyhedron if and only if $3(c_1)^2 \neq 2(c_2)^3$.



4. Contibutions to the topological zeta function and the A´Campo monodromy zeta function of the trinomial forms

Proposition 4.1. *The monodromy zeta function and the local topological zeta function $Z(F,s)_0$ of F equal*

$$(1-t^2)^{\alpha}\prod_I \zeta_{f_I}(t)^{-\chi(E_I^\circ)}$$

$$\sum_I \chi(E_I^\circ)Z(f_I,\omega_I)_{o_I}$$

where I varies over the statification associated to the resolution π of $b_2b_3D_zF$, the exponent α is the number of points of blow-ups needed to obtain π, and o_I is a point in E_I°.

Example 4.2.

4.1. Trinomial forms for strata E_i°.

4.1.1. *First case $f=z^3-x^{m_3}$ and $\omega=x^{\nu_1-1}dxdz$.* Newton polyhedra has three compact faces: the 0-dimensional faces τ_0 corresponding to the monomial z^3 and τ_1 corresponding to the monomial x^{m_3}, and the 1-dimensional face σ contained in

$$3p+m_3r=3m_3$$

The corresponding candidate pole is

$$3m_3s+3\nu_1+m_3$$

The topological zeta function $Z_{\text{top}}(f,\omega,s)_0$ equals

$$\frac{3\nu_1+m_3+(3\nu_1+m_3-\nu_1(m_3,3)^2)s}{\nu_1(3m_3s+3\nu_1+m_3)(s+1)}$$

or equivalently $J_{\check\tau_0}-\frac{(3,m_3)J_{\check\sigma}s}{s+1}+J_{\check\tau_1}$ with the terms $J_{\check\tau_0}$, $J_{\check\tau_1}$ and $J_{\check\sigma}$ given as follows.

(1) The dual cone $\check\tau_1$ to the face τ_1 is simplicial and generated by the rays

$$\frac{1}{(m_3,3)}(3,m_3),e_2.$$

This cone has multiplicity $\frac{3}{(m_3,3)}$. Its contribution to the local topological zeta function is

$$J_{\check\tau_1}=\frac{3}{\nu_1(3m_3s+3\nu_1+m_3)}.$$

(2) The dual cone $\check\sigma$ of the face σ is the simplical cone generated by the ray

$$\frac{1}{(m_3,3)}(3,m_3).$$

Its contribution to the local zeta function is

$$J_{\check\sigma}=\frac{(3,m_3)}{(3m_3s+3\nu_1+m_3)}.$$

(3) The dual cone $\check{\tau}_0$ to the face τ_0 is the cone generated by the rays

$$\frac{1}{(m_3,3)}(3,m_3), e_1.$$

This cone has multiplicity $\frac{m_3}{(m_3,3)}$. Its contribution to the the local topological zeta function is

$$J_{\check{\tau}_0} = \frac{m_3}{(3m_3 s + 3\nu_1 + m_3)}.$$

Similarly we have

$$\zeta_f(t) = \frac{(1-t^3)(1-t^{m_3})}{(1-t^{3m_3})^{(m_3,3)}}$$

4.1.2. *Second case* $f = z^3 - x^{m_2}z - x^{m_3}$ *and* $\omega = x^{\nu_1 - 1}dxdz$ *such that* $3m_2 < 2m_3$. Newton polyhedra has five compact faces: the 0-dimensional faces τ_0 corresponding to the monomial z^3, τ_1 corresponding to the monomial $x^{m_2}z$, and τ_2 corresponding to the monomial x^{m_3}, and the 1-dimensional faces σ_1 contained in the hyperplane

$$2p + m_2 r = 3m_2$$

and σ_2 contained in the hyperplane

$$p + (m_3 - m_2)r = m_3$$

The candidate poles are

$$3m_2 s + 2\nu_1 + m_2 \quad \text{and} \quad m_3 s + \nu_1 + m_3 - m_2$$

The topological zeta function $Z_{\text{top}}(f,\omega,s)_0$ equals

$$\frac{(2 - \nu_1(m_2,2)^2)s + 2(\nu_1+1)}{(3m_2 s + 2\nu_1 + m_2)(s+1)}$$

or equivalently $J_{\check{\tau}_0} - (m_2,2)J_{\check{\sigma}_1}\frac{s}{s+1} + J_{\check{\tau}_1} - \frac{J_{\check{\sigma}_2}s}{s+1} + J_{\check{\tau}_2}$ with the terms $J_{\check{\tau}_0}$, $J_{\check{\tau}_1}$, $J_{\check{\tau}_2}$, $J_{\check{\sigma}_1}$ and $J_{\check{\sigma}_2}$ given as follows. Note that the candidate pole $m_3 s + \nu_1 + m_3 - m_2$ is a false candidate pole.

(1) The dual cone $\check{\tau}_2$ to the face τ_2 is the regular cone generated by the rays

$$(1, m_3 - m_2), e_2$$

give the contribution

$$J_{\check{\tau}_2} = \frac{1}{(m_3 s + \nu_1 + m_3 - m_2)}$$

(2) The dual cone $\check{\sigma}_2$ of the face σ_2 is the regular cone generated by the rays

$$(1, m_3 - m_2)$$

give the contribution

$$J_{\check{\sigma}_2} = \frac{1}{(m_3 s + \nu_1 + m_3 - m_2)}$$

(3) The dual cone $\check{\tau}_1$ to the face τ_1 is the simplicial cone generated by the rays

$$(1, m_3 - m_2), \frac{1}{(m_2, 2)}(2, m_2)$$

The multiplicity of this cone is $\frac{m_2+2m_3}{(2,m_2)}$. Its contribution to the the local topological zeta function is

$$J_{\check{\tau}_1} = \frac{3m_3 - 2m_2}{(m_3 s + \nu_1 + m_3 - m_2)(3m_2 s + 2\nu_1 + m_2)}$$

(4) The dual cone $\check{\sigma}_1$ of the face σ_1 is the cone generated by the rays

$$\frac{1}{(n_2, 2)}(2, m_2)$$

It gives the contribution

$$J_{\check{\sigma}_1} = \frac{(m_2, 2)}{(3m_2 s + 2\nu_1 + m_2)}$$

(5) The dual cone $\check{\tau}_0$ of the face τ_0 is the simplicial cone generated by the rays

$$e_1, \frac{1}{(m_2, 2)}(2, m_2)$$

The multiplicity of this cone is $\frac{2}{(2,m_2)}$. Its contribution to the the local topological zeta function is

$$J_{\check{\tau}_0} = \frac{2}{\nu_1(3m_2 s + 2\nu_1 + m_2)}$$

Similarly we have

$$\zeta_f(t) = \frac{(1 - t^3)}{(1 - t^{3m_2})^{(m_2,2)}}$$

We remark that this case it is equivalent to $f = z(z^2 - x^{m_2})$.

4.1.3. *Third case* $f = z^3 - x^{m_1}z^2 - x^{m_3}$ *and* $\omega = x^{\nu_1 - 1}dxdz$ *such that* $m_3 > m_1$. Newton polyhedra has five compact faces: the 0-dimensional faces τ_0 corresponding to the monomial z^3, τ_1 corresponding to the monomial $x^{m_1}z^2$, and τ_2 corresponding to the monomial x^{m_3}, and the 1-dimensional faces σ_1 contained in the hyperplanes

$$p + m_1 r = 3m_1$$

and σ_2 contained in the hyperplanes

$$2p + (m_3 - m_1)r = 2m_3$$

The candidate poles are

$$3m_1 s + \nu_1 + m_1 \quad \text{and} \quad 2m_3 s + 2\nu_1 + m_3 - m_1$$

The topological zeta function $Z_{\text{top}}(f, \omega, s)_0$ equals

$$\frac{(2m_1m_3 + 6m_1\nu_1 - 2m_3\nu_1 - 3m_1\nu_1c^2)s^2 + (3m_1m_3 - m_1^2 + 8m_1\nu_1 - \nu_1^2c^2s - m_1\nu_1c^2)s - m_1^2 + m_1m_3 + m}{(s+1)(3m_1s + \nu_1 + m_1)(2m_3s + 2\nu_1 + m_3 - m_1)}$$

or equivalently $V_0 - \frac{As}{s+1} + V_1 - \frac{A_2 s}{s+1} + V_2$ with the terms V_0, V_1 and A given as follows.

(1) The dual cone $\check{\tau}_2$ to the face τ_2 is the regular cone generated by the rays

$$e_2, \frac{1}{(m_3-m_1,2)}(2, m_3-m_1)$$

give the contribution

$$J_{\check{\tau}_2} = \frac{2}{(2m_3s+2\nu_1+m_3-m_1)}$$

(2) The dual cone $\check{\sigma}_2$ of the face σ_2 is the regular cone generated by the rays

$$\frac{1}{(m_3-m_1,2)}(2, m_3-m_1)$$

give the contribution

$$J_{\check{\sigma}_2} = \frac{(m_3-m_1,2)}{(2m_3s+2\nu_1+m_3-m_1)}$$

(3) The dual cone $\check{\tau}_1$ to the face τ_1 is the simplicial cone generated by the rays

$$\frac{1}{(m_3-m_1,2)}(2, m_3-m_1), (1, m_1)$$

The multiplicity of this cone is $\frac{m_2+2m_3}{(2,m_2)}$. Its contribution to the the local topological zeta function is

$$J_{\check{\tau}_1} = \frac{m_3-3m_1}{(2m_3s+2\nu_1+m_3-m_1)(3m_1s+\nu_1+m_1)}$$

(4) The dual cone $\check{\sigma}_1$ of the face σ_1 is the cone generated by the ray

$$(1, m_1)$$

It gives the contribution

$$J_{\check{\sigma}_1} = \frac{1}{(3m_1s+\nu_1+m_1)}$$

(5) The dual cone $\check{\tau}_0$ of the face τ_0 is the simplicial cone generated by the ray

$$e_1, (1, m_1)$$

The multiplicity of this cone is $\frac{2}{(2,m_2)}$. Its contribution to the the local topological zeta function is

$$J_{\check{\tau}_0} = \frac{m_1}{\nu_1(3m_1s+\nu_1+m_1)}$$

Similarly we have

$$\zeta_f(t) =$$

We remark that this case is equivalent to $f = (z - x^{m_1})(z^2 - x^{m_3-m_2})$

4.2. **Trinomial forms for strata $E^\circ_{i,j}$.**

4.2.1. *First case* $f = z^3 - x^{m_3}y^{n_3}$ *and* $\omega = x^{\nu_1-1}y^{\nu_2-1}dxdydz$. Newton polyhedra has three compact faces: the 0-dimensional faces τ_0 corresponding to the monomial z^3 and τ_1 corresponding to the monomial $x^{m_3}y^{n_3}$, and the 1-dimensional face σ contained in

$$3p + m_3r = 3m_3, \qquad 3q + n_3r = 3n_3$$

The corresponding candidate poles are

$$3m_3s + 3\nu_1 + m_3, \qquad 3n_3s + 3\nu_2 + n_3$$

The topological zeta function $Z_{\text{top}}(f, \omega, s)_0$ equals

$$\frac{((3s+1)m_3n_3 + 3(\nu_1n_3 + \nu_2m_3 + 3\nu_1\nu_2))(s+1) - 3\nu_1\nu_2(3, m_3, n_3)^2s}{\nu_1\nu_2(3m_3s + 3\nu_1 + m_3)(3n_3s + 3\nu_2 + n_3)(s+1)}$$

or equivalently $V_0 - \frac{(3,m_3,n_3)As}{s+1} + V_1$ with the terms V_0, V_1 and A given as follows.

(1) The dual cone $\check{\tau}_1$ to the face τ_1 is simplicial and generated by the rays

$$\frac{1}{(m_3, 3)}(3, 0, m_3), \frac{1}{(n_3, 3)}(0, 3, n_3), e_3.$$

This cone has multiplicity $\frac{9}{(m_3,3)(n_3,3)}$. Its contribution to the local topological zeta function is

$$V_1 = \frac{9}{(3m_3s + 3\nu_1 + m_3)(3n_3s + 3\nu_2 + n_3)}.$$

(2) The dual cone $\check{\sigma}$ of the face σ is the simplical cone generated by the rays

$$\frac{1}{(m_3, 3)}(3, 0, m_3), \frac{1}{(n_3, 3)}(0, 3, n_3).$$

This cone has multiplicity $\frac{3(m_3,n_3,3)}{(m_3,3)(n_3,3)}$. Its contribution to the local zeta function is

$$A = \frac{3(m_3, n_3, 3)}{(3m_3s + 3\nu_1 + m_3)(3n_3s + 3\nu_2 + n_3)}.$$

(3) The dual cone $\check{\tau}_0$ to the face τ_0 is the cone generated by the rays

$$\frac{1}{(m_3, 3)}(3, 0, m_3), \frac{1}{(n_3, 3)}(0, 3, n_3), e_1, e_2$$

which is not simplicial. Using the expression $\check{\tau}_0 = \mathbb{R}^3_{\geq 0} - \check{\tau}_1 - \check{\sigma}$, its contribution to the the local topological zeta function is

$$V_0 = \frac{1}{\nu_1\nu_2(3s+1)} - \frac{9}{(3m_3s + 3\nu_1 + m_3)(3n_3s + 3\nu_2 + n_3)(3s+1)}$$

$$= \frac{m_1m_2(3s+1) + 3(\nu_1m_2 + \nu_2m_1)}{\nu_1\nu_2(3m_3s + 3\nu_1 + m_3)(3n_3s + 3\nu_2 + n_3)}$$

4.3. Second case $f = z^3 - x^{m_2}y^{n_2}z - x^{m_3}y^{n_3}$ and such that $m_2 \leq 2m_3, 3n_2 \leq 2n_3$ and $3(m_2, n_2) \neq 2(m_3, n_3)$.

Newton polyhedra has five compact faces: the 0-dimensional faces τ_0 corresponding to the monomial z^3, τ_1 corresponding to the monomial $x^{m_2}y^{n_2}z$, and τ_2 corresponding to the monomial $x^{m_3}y^{n_3}$, and the 1-dimensional faces σ_1 contained in the hyperplanes

$$2p + m_2r = 3m_2, \qquad 2q + n_2r = 3n_2$$

and σ_2 contained in the hyperplanes

$$p + (m_3 - m_2)r = m_3, \qquad q + (n_3 - n_2)r = n_3$$

The candidate poles are

$$3m_2s + 2\nu_1 + m_2, \qquad 3n_2s + 2\nu_2 + n_2$$

and

$$m_3s + \nu_1 + m_3 - m_2, \qquad n_3s + \nu_2 + n_3 - n_2$$

The topological zeta function is the sum of the following five terms

(1) The dual cone $\check{\tau}_2$ to the face τ_2 is the regular cone generated by the rays

$$(1, 0, m_3 - m_2), (0, 1, n_3 - n_2), e_3$$

give the contribution

$$\frac{1}{(m_3s + \nu_1 + m_3 - m_2)(n_3s + \nu_2 + n_3 - n_2)}$$

(2) The dual cone $\check{\sigma}_2$ of the face σ_2 is the regular cone generated by the rays

$$(1, 0, m_3 - m_2), (0, 1, n_3 - n_2)$$

give the contribution

$$\frac{-s}{(m_3s + \nu_1 + m_3 - m_2)(n_3s + \nu_2 + n_3 - n_2)(s + 1)}$$

(3) The dual cone $\check{\tau}_1$ to the face τ_1 is the cone generated by the rays

$$(1, 0, m_3 - m_2), (0, 1, n_3 - n_2), \frac{1}{(m_2, 2)}(2, 0, m_2), \frac{1}{(n_2, 2)}(0, 2, n_2)$$

which is not simplicial. Denote by ρ the cone generated by $\frac{1}{(m_2,2)}(2, 0, m_2)$, $\frac{1}{(n_2,2)}(0, 2, n_2)$ and e_3. Using the expression $\check{\tau}_1 = \rho - \check{\tau}_2 - \check{\sigma}_2$, its contribution to the the local topological zeta function is

$$\frac{4}{(m_2, 2)(n_2, 2)(3m_2s + 2\nu_1 + m_2)(3n_2s + 2\nu_2 + n_2)} - \frac{1}{(m_3s + \nu_1 + m_3 - m_2)(n_3s + \nu_2 + n_3 - n_2)}$$

(4) The dual cone $\check{\sigma}_1$ of the face σ_1 is the cone generated by the rays

$$\frac{1}{(n_2, 2)}(2, 0, m_2), \frac{1}{(n_2, 2)}(0, 2, n_2)$$

This cone has multiplicity $2(m_2, n_2, 2)$. It gives the contribution

$$\frac{-2(m_2, n_2, 2)s}{(3m_2s + 2\nu_1 + m_2)(3n_2s + 2\nu_2 + n_2)(s + 1)}$$

(5) The dual cone $\check{\tau}_0$ of the face τ_0 is the cone generated by the rays

$$e_1, e_2, \frac{1}{(m_2,2)}(2,0,m_2), \frac{1}{(n_2,2)}(0,2,n_2)$$

Using the expression $\check{\tau}_0 = \mathbb{R}^3_{\geq 0} - \rho - \check{\sigma}_2$, its contribution to the the local topological zeta function is

$$\frac{1}{\nu_1\nu_2(3s+1)} - \frac{2}{(3m_2s+2\nu_1+m_2)(3n_2s+2\nu_2+m_2)(3s+1)}$$

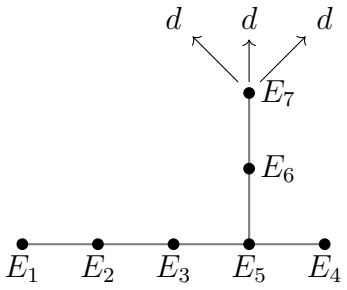
4.4. Third case $f = z^3 - x^{m_1}y^{n_1}z^2 - x^{m_3}y^{n_3}$ and such that $3m_1 \leq m_3, 3n_1 \leq n_3$ and $3(m_1,n_1) \neq 2(m_3,n_3)$.

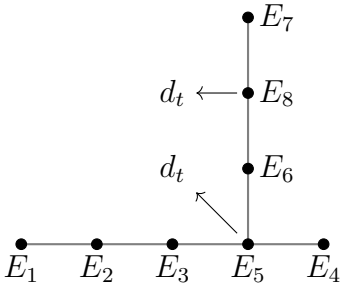
5. On generic projections

Example 5.1. Consider the Weierstrass polynomial $F = z^3 - 3f(x,y)z + 2g(x,y)$ with $f = y^2 - x^7 + x^5y$ and $g = f^2 + (y+x^2)^6$. It defines a germ of isolated surface singularity with Milnor number $\mu = 41$. The discriminant of F is given by

$$f^3 - g^2 = f^3 - f^4 - 2f^2(y+x^2)^6 + (y+x^2)^{12}$$

and it has the topological type given by three $(2,7)$-cusp with pairwise intersection multiplicity 16. Its dual resolution graph is pictured (left) bellow.





The polinomial $F_t = z^3 - 3f_t(x,y)z - 2g_t(x,y)$ with

$$f_t = f + t^2y^2 \qquad \text{and} \qquad g_t = g + \frac{3}{2}tyf + t^3y^3$$

defines a μ-constant topological deformation of F whose discriminant $f_t^3 - g_t^2$ has the following embedded topological type. The discriminant of F_t has two branches. One is a $(2,7)$-cusp and the other branch has two Puiseux pairs, namely, $(2,7)$ and $(2,3)$. Their intersection multiplicity is 28. The dual resolution graph of the discriminant is pictured (right) above.

The topological zeta function and the monodromy zeta function of F and F_t are given by

$$\frac{-41}{168}\frac{s^2 - 283/123s - 304/123}{(s+16/21)(s+19/24)(s+1)} \quad \text{and} \quad \Phi_1\Phi_2\Phi_3\Phi_4\Phi_6\Phi_7\Phi_8\Phi_{12}\Phi_{21}\Phi_{24}$$

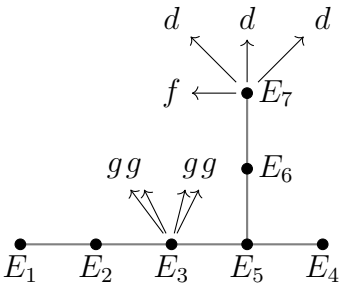
In particular, they both have Milnor number equal to 41. This can be computed from embedded resolutions of the meromorphic functions $\mathcal{M}$ and $\mathcal{M}_t$ with the following numerical

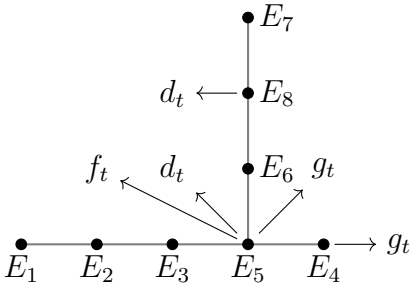
data

	E_1	E_2	E_3	E_4	E_5	E_6	E_7
f	2	4	6	7	14	15	16
g	4	8	12	12	24	24	24
d	6	12	18	21	42	45	48

	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8
f_t	2	4	6	7	14	14	14	28
g_t	3	6	9	11	21	21	21	42
d_t	6	12	18	21	42	44	45	90

and dual resolution graphs





In order to that there check that both F and F_t define the same embedded topological type, notice that

$$F_t(x, y, z + yt) = F(x, y, t) + 3tyz^2$$

Moreover, in the left-hand side change $z \mapsto z+ty$ is an isomorphism that does not change the discriminant with respect to z, nor the Milnor number. Finally, in the right-hand side we have a μ-constant deformation of isolated hypersurface singularities of the form $f(x) + tg(x)$, that preserves the embedded topological type according to [**?**].

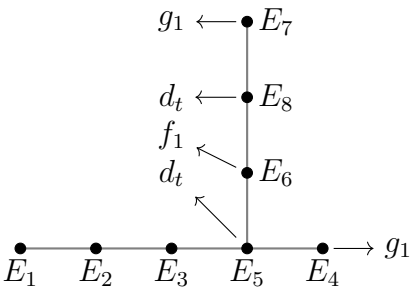
Finally, it is worth mentioning that the topological type of the discriminant of F_t can be realizable as $f_1^3 - g_1^2$ for another pair of functions (f_1, g_1). The corresponding function $F_1 = z^3 - 3f_1z + 2g_1$ does not belong to the embedded topological type equivalence class of F and F_t because their Milnor number and monodromy zeta function do not coincide. The topological and monodromy zeta function of F_1 are

$$\frac{-5}{21}\frac{s - 16/5}{(s + 16/21)(s + 1)} \qquad \text{and,} \qquad \Phi_1\Phi_3\Phi_7^2\Phi_{21}^2$$

In particular, the Milnor number of F_1 is 38. This can be computed from embedded resolution of the meromorphic functions $\mathcal{M}_1$ with the following numerical data

	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8
f_t	2	4	6	7	14	15	15	30
g_t	3	6	9	11	21	22	23	45
d_t	6	12	18	21	42	44	45	90

and dual resolution graph.


g_1
E_7
d_t
E_8
f_1
d_t
E_6
g_1
E_1
E_2
E_3
E_5
E_4