STEENBRINK'S MHS WITH NON ISOLATED SINGULARITIES

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Steenbrink [3] constructed a MHS on the vanishing cohomology of an isolated singular point of a hypersurface. We study the application of Steenbrink's construction to the case of non isolated singularities.

1. General setting.

Consider $P \in \mathbb{C}[z_0, \ldots, z_n]$ such that P is non constant and $P(o) = 0$. Denote the set of zeros of P by $V(P)$, i.e., $V(P) = \{z \in \mathbb{C}^{n+1} \,:\, P(z) = 0\}$. Assume that o is a (non isolated) singular point of $V(P)$.

Let d be the degree of the polynomial P, and denote by P_h the homogeneous polynomial determined by P, i.e., $P_h(z_0 : \cdots : z_{n+1}) = z_{n+1}^d P(z_0/z_{n+1}, \ldots, z_n/z_{n+1})$. Denote by $\bar{P}$ the map from $\mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^1 = \mathbb{C} \cup \infty$ given by P_h/z_{n+1}^d. Let $\rho : W \to \mathbb{P}^{n+1}$ be the resolution of the indeterminacies of $\bar{P}$ that are obtained by means of blow ups with center at the hyperplane at infinity. Notice that $\bar{P}$ is well defined at the point $\bar{o} := [o : 1] \in \mathbb{P}^{n+1}$. Therefore we can consider $\bar{o} \in W$. Let S a small disk about $0 \in \mathbb{C} \cup \infty$ and let $Y := (\rho \circ \bar{P})^{-1}(S)$ and let $f_1 := (\rho \circ \bar{P})_{|Y}$. The disk has be chosen small enought such that the value $0 \in \mathbb{C} \cup \infty$ is the unique critical value of the map f_1; as it is proper the fibers $Y_t = f_1^{-1}(t)$ are smooth and projective for $t \neq 0$.

Finally, denote by $f : X \to S$ the embedded resolution of the map f_1. It is obtained by a modification $\psi : X \to Y$, i.e., $f = f_1 \circ \psi$. Denote the fibers of f by X_t, i.e., $X_t = f^{-1}(t)$ for $t \in S$. The fibers X_t are isomorphic to Y_t, for $t \neq 0$. The central fiber $X_0 = f^{-1}(0)$ is a simple normal crossing divisor on X while Y_0 had more complicated singularities.

$$\begin{array}{ccccccc}
\tilde{X} & \xrightarrow{\pi} & X & \xrightarrow{\psi} & Y & \xrightarrow{\rho} & \mathbb{P}^{n+1} \\
\downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_1} & & \vdots{\scriptstyle \bar{P}} \\
\tilde{S} & \longrightarrow & S & \xrightarrow{=} & S & \xrightarrow{i} & \mathbb{P}^1
\end{array}$$

Figure 1.

The irreducible components of X_0 can be classified into three types. Let us denote by E' the set of irreducible components E_i of X_0 that are exceptional and are mapped to $\bar{o} \in \mathbb{P}^{n+1}$ under $\rho \circ \psi$ or, in other words, $\operatorname{Supp} E_i \subset (\rho \circ \psi)^{-1}(\bar{o})$. Let us denote by E'' the set of irreducible components E_j of X_0 such that such that $\bar{o}$ is strictly contained in $(\rho \circ \psi)(E_j)$. In particular, the irreducible components of the strict transform of $V(P)$ correspond to irreducible components in E''. Finally, let us denote by E''' the set of irreducible components E_k of X_0 such that $\bar{o} \notin (\rho \circ \psi)(E_k)$. Since $\bar{o}$ plays a special role, we will assume an extra hypothesis, namely that the preimage of $\bar{o}$ is contained in E'.

Remark 1.1. Steenbrink asumes that o is an isolated singular point of $V(P)$. Under this hypothesis Brieskorn constructed an irreducible homogeneous polynomial P' such that

$V(P)$ has a unique singular point which is analitically equivalent to $(V(P), o)$. Using P', Steenbrink can assume that E'' is a prime divisor (the strict transform of $V(P')$), and that E''' is empty. The extra hypothesis is always fullfilled in the isolated singularity case.

Let e be a common multiple of the multiplicities of the divisor E. Consider the covering $\sigma : \tilde{S} \to S$ given by $\sigma(t) = t^e$ with $\tilde{S}$ the disk centered in $0 \in \mathbb{C}$ and radius $\eta^{1/e}$. Consider $\tilde{X}$ the normalization of the fibered product $X \times_S \tilde{S}$ and let $\tilde{f} : \tilde{X} \to \tilde{S}$ and $\pi : \tilde{X} \to X$ the natural maps. Denote by D the divisor $\pi^{-1}(X_0)$. A decomposition into D', D'', D''' is done as for E.

Consider also the universal covering of the punctured disk $S^* := S \setminus \{0\}$ given the map $H := \{z \in \mathbb{C} : \Im(z) > 0\} \to S^*$ defined as $z \mapsto \exp(2\pi i z)$. Denote by X_∞ the fibered product $X \times_S H$.

Let $A^\bullet$ be the complex of sheaves on the divisor D that defines, according to Schmid [1], in the unipotent case, and Steenbrink [2, 3], a limit Hodge structure on the cohomology groups $H^p(X_\infty)$ associated with the fibration $f_{|X \setminus X_0} : X \setminus X_0 \to S \setminus \{0\}$.

Let B a small ball in Y with center $\tilde{o}$ and radius ϵ. Assume that all balls centered at $\tilde{o}$ and with radius less than ϵ intersect Y_0 transversally. Choose $\eta \ll \epsilon$ such that all fibers with Y_t with $|t| < \eta$ intersect B transversally. Denote the intersection $Y_t \cap B$ by B_t. Under the above hypothesis for ϵ and η, the complex manifold B_t with $t \neq 0$ is diffeomorphic to the Milnor fiber of P at o.

Denote by B_∞ the fibered product $B \times_S H$. Our first aim is to compute $H^p(B_\infty)$ by means of a complex of sheaves $B^\bullet$ on the divisor D with support on D'. This complex should satisfy the following additional properties:

- The complex $B^\bullet$ will be defined as a locally free on the abstract strata
$$\check{D}_k := \bigsqcup_{I, |I|=k} \bigcap_{i \in I} D_i$$
consisting on disjoint unions of intersections of exactly k irreducible components of D.
- Given the natural map $i_k : \check{D}_k \to D$, the direct image $(i_k)_* B^\bullet$ is locally free too.
- These properties are also satisfied by $A^\bullet$. Actually, if $\tilde{f}(D_I) = \bar{o}$ then $B^\bullet_{|D_I} = A^\bullet_{|D_I}$; otherwise $B^\bullet_{|D_I} = 0$.

Using the above hypothesis and the spectral sequence associated with the complex $B^\bullet$ we will show that $H^q(B_\infty, \mathbb{Q}) = \mathbb{H}^p(D, B^\bullet_{\mathbb{Q}})$.

Remark 1.2. If I am not wrong the weight filtration in the spectral sequence is obtained gluing several weight filtrations for the logarithm of the unipotent part of the monodromy centered at $n, n+1, \ldots, n+k$, where $k < leqr + 1$, if r is the dimension of the singular locus.

References

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