$$\frac{S+1}{S(NS+1)} = \frac{S+1}{NS\left(S+\frac{1}{N}\right)}$$

$$\hat{Z}_{top}(F,S) = \frac{S+1}{S}\, Z_{bp}(F,S)$$

$$\hat{Z}_{top}(F,S) = \frac{S+1}{NS\left(S+\frac{1}{N}\right)} + \frac{N-1}{N}\hat{Z}_{bp}\left(\beta, S+\frac{1}{N}\right) - \tilde{Z}_{bp}\left(\beta, S+\frac{1}{N}\right)$$

$$Z^{(e)}_{bp}(F,S) = \frac{1}{N\left(S+\frac{1}{N}\right)} - \frac{1}{N}\hat{Z}_{bp}\left(\beta, S+\frac{1}{N}\right) - \tilde{Z}_{top}\left(\beta, S+\frac{1}{N}\right)$$

$$\hat{Z}_{top}(F,S) = \frac{(S+1)}{NS\left(S+\frac{1}{N}\right)} - \frac{1}{N\left(S+\frac{1}{N}\right)}\hat{Z}_{top}\left(\beta, S+\frac{1}{N}\right) + \hat{Z}_{bp}\left(\beta, S+\frac{1}{N}\right) - \tilde{Z}\left(\beta, S+\frac{1}{N}\right)$$

$$Z^{(e)}_{top}(F,S) = \frac{1}{N\left(S+\frac{1}{N}\right)} - \frac{S}{N\left(S+\frac{1}{N}\right)(S+1)}\hat{Z}_{top}\left(\beta, S+\frac{1}{N}\right) + Z^{(e)}_{bp}\left(\beta, S+\frac{1}{N}\right) - \tilde{Z}\left(\beta, S+\frac{1}{N}\right)$$

$$\tilde{Z}(\beta,S) = \frac{1}{N}\sum_{e} \sigma_2(e)\, Z^{(e)}(\beta,S) - \frac{1}{N}\hat{Z}_1(\beta,S)$$

$$\tilde{Z}\left(\beta, S+\frac{1}{N}\right) = \frac{1}{N}\sum_{1/e/N} \mu - \frac{1}{N}\mu(N)\, Z^{[1]}$$

$$f(e) = Z_{top}^{\left[\frac{N}{e}\right]} \qquad g(e) = Z_{top}^{[e]}$$

$$f(e) = \sum_{1/\ell/e} g(\ell)$$

$$f(e) = Z_{top}^{\left(\frac{N}{e}\right)} = \sum_{N/e/\frac{N}{e}/N} Z_{top}^{\left[\frac{N}{e}\right]} = \sum_{1/\ell_1/e} Z_{top}^{[\ell_1]} = \sum_{1/\ell_1/e} g(e)$$

$$\ell_1 = \frac{N}{\ell}$$

$$g(e) = \sum_{1/\ell/e} \mu\left(\frac{e}{\ell}\right) f(\ell) = \sum_{\ell_1/e} \mu(\ell_1) f\left(\frac{e}{\ell_1}\right)$$

$$\ell_1 = \frac{e}{\ell}$$

$$Z_{top}^{[e]} = \sum_{e/\ell/N} \mu\left(\frac{\ell}{e}\right) Z_{top}^{(\ell)}$$