COMPUTATION OF $\mathcal{A}_e$-CODIMENSION

1. Theoretical framework

1.1. **The $\mathcal{A}_e$-tangent over the target.** Consider a polynomial map-germ $f\colon (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$, with coordinate functions $f_j(x)$, such that the map given by the first n coordinates functions,

$$\overline{f}\colon (\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$$

is finite. To avoid confusion, we write

$$X = (\mathbb{C}^n,0) \quad \text{and} \quad \overline{f}\colon X \to (\mathbb{C}^n,0).$$

Let

$$g_1,\ldots,g_m \in \overline{f}_*\mathcal{O}_n$$

be a minimal set of generators of $\overline{f}_*\mathcal{O}_n$, which is the ring $\mathcal{O}_X$, regarded as an $\mathcal{O}_n$-algebra via composition with $\overline{f}$. From the fact that X is Cohen-Macaulay and $\dim X = n$ it follows that there is an isomorphism

$$\bigoplus^m \mathcal{O}_n \xrightarrow{\varphi} \overline{f}_*\mathcal{O}_X,$$

given by

$$(H^k)_{k=1}^m \mapsto \sum_{k=1}^m H^k \cdot g_k,$$

where $H^k \cdot g_k = (H^k \circ \overline{f})\, g_k$.

Remark 1.1. All our troubles have to do with the behaviour of the inverse of φ. Assuming $\overline{f}$ to be a polynomial mapping and the generators g_k to be polynomials, the isomorphism φ takes polynomial vectors to polynomial functions. However, the inverse φ^{-1} need not take polynomials to polynomial vectors. A simple example is provided by $\overline{f}(x) = x + x^2$, with a single generator $g_1 = 1$. The function $h(x) = x$ cannot be expressed as a polynomial in $x + x^2$.

Similarly, consider the the $\mathcal{O}_n$-module

$$\theta(f) = \bigoplus^p \overline{f}_*\mathcal{O}_X$$

and the isomorphism

$$\bigoplus^p \bigoplus^m \mathcal{O}_n \xrightarrow{\psi} \overline{f}_*\mathcal{O}_X,$$

given by

$$(H_j^k)_{\substack{1\le k\le m\\ 1\le j\le p}} \mapsto (\sum_{k=1}^m H_j^k \cdot g_k)_{1\le j\le p},$$

Finally consider the $\mathcal{A}_e$-tangent submodule

$$T\mathcal{A}_e(f) = \left\langle \frac{\partial f}{\partial x_i} \mid 1 \leq i \leq n \right\rangle_{\mathcal{O}_X} + \bigoplus^p f^*\mathcal{O}_p,$$

where $f^*\mathcal{O}_p$ stands for the subring of $\mathcal{O}_n$ consisting of functions of the form $f^*H = H \circ f$, with $H \in \mathcal{O}_p$ (Caution: composition is taken with f and not with $\overline{f}$).

Proposition 1.2. *As an $\mathcal{O}_n$-module, the space $T\mathcal{A}_e(f)$ is generated by the following collection of vectors*

- $\dfrac{\partial f}{\partial x_i}\, g_k$, *with* $1 \leq i \leq n$, $1 \leq k \leq m$
- $u_j^\alpha = (0, \ldots, \overset{j}{\smile}{m_\alpha}, \ldots, 0)$, *with* $1 \leq j \leq p$, $\alpha = (\alpha_{n+1}, \ldots, \alpha_p)$ *and* $0 \leq \alpha_\alpha \leq m$, *and where*

$$m_\alpha = f_{n+1}^{\alpha_{n+1}} \ldots f_p^{\alpha_p}$$

Remark 1.3. Question: is it enough to take $\alpha = (\alpha_{n+1}, \ldots, \alpha_p)$ with $0 \leq |\alpha| \leq m$ instead?

1.2. **The $\mathcal{A}_e$-codimension.** By definition, the $\mathcal{A}_e$-codimension of f is

$$\operatorname{codim}_{\mathcal{A}_e}(f) = \dim_{\mathbb{C}} \frac{\theta(f)}{T\mathcal{A}_e(f)}$$

The difficulty in computing the $\mathcal{A}_e$-codimension resides in the fact that it is a subset of $\theta(f)$, thus it consists of vectors in $\mathcal{O}_X$, but it is not a $\mathcal{O}_X$ module. To solve this, we may move everything isomorphically to $\bigoplus^p \bigoplus^m \mathcal{O}_n$, applying the isomorphism ψ, so that the calculation that we actually perform is

$$\operatorname{codim}_{\mathcal{A}_e}(f) = \dim_{\mathbb{C}} \frac{\bigoplus^p \bigoplus^m \mathcal{O}_n}{\psi^{-1}(T\mathcal{A}_e(f))}$$

Since ψ is an isomorphism of $\mathcal{O}_n$-modules, the generators of $\psi^{-1}(T\mathcal{A}_e(f))$ are

- $\psi^{-1}\left(\dfrac{\partial f}{\partial x_i}\, g_k\right)$, with $1 \leq i \leq n$, $1 \leq k \leq m$
- $\psi^{-1}(u_j^\alpha) = (0, \ldots, \varphi^{-1}(m_\alpha), \ldots, 0)$, with $1 \leq j \leq p$, $\alpha = (\alpha_{n+1}, \ldots, \alpha_p)$ and $0 \leq \alpha_\alpha \leq m$, and where

$$m_\alpha = f_{n+1}^{\alpha_{n+1}} \ldots f_p^{\alpha_p}$$

Remark 1.4. Observe that, in order to compute $\psi^{-1}(u_j^\alpha)$, it is enough to compute $\varphi^{-1}(f_\ell)$, for $n+1 \leq \ell \leq p$

1.3. **Computing $\mathcal{A}_e$-codimension by approximations.** The obstacle to compute $\mathcal{A}_e$-codimension is that we must compute the generators

$$\psi^{-1}(\frac{\partial f}{\partial x_i}\, g_k) \quad \text{and} \quad \psi^{-1}(u_j^\alpha),$$

which may even fail to have polynomial entries. This may be solved by introducing a decreasing filtration of $\bigoplus^p \bigoplus^m \mathcal{O}_n$.

Definition 1.5. A *complete, separated, origin supported filtration* of an $\mathcal{O}_n$ module M is a sequence

$$M_0 \supseteq M_1 \supseteq \cdots \supseteq M_r \supset \dots$$

of $\mathcal{O}_n$-modules, satisfying the following properties.

(1) Completeness: $M_0 = M$.
(2) Separation: $\bigcap_{r=1}^{\infty} M_r = 0$.
(3) Origin supported: $\operatorname{supp}(M_r/M_{r+1}) \subseteq \{0\}$, for all r.

From now on, we just use the term filtration, and always assume them to be complete, separated and origin supported.

Remark 1.6. Completeness is easily achieved, by adding an extra initial term $M_0 = M$ to any given filtration.

Separation may be though of as the property that ensures that the approximations that we will introduce later on actually converge to the desired result.

The origin supported condition is equivalent to the fact that $\operatorname{supp}(M/M_{r+1}) \subseteq \{0\}$ (By induction, the stalks at $x \neq 0$ of M_0 and M_r must agree). For the module $M = \bigoplus^p \bigoplus^m \mathcal{O}_n$ that we study, this is in turn equivalent to the fact that, for all r, there exists K, such that $\bigoplus^p \bigoplus^m \mathfrak{m}_n^K \subseteq M_r$, where $\mathfrak{m}_n$ stands for the maximal ideal in $\mathcal{O}_n$).

Definition 1.7. Given a filtration $M_\bullet$ of $\bigoplus^p \bigoplus^m \mathcal{O}_n$, we define

$$d_r = \dim_{\mathbb{C}} \frac{\psi^{-1}(T\mathcal{A}_e(f)) + M_r}{\psi^{-1}(T\mathcal{A}_e(f)) + M_{r+1}} < \infty$$

(Observe that the fact that $d_r < \infty$ is a direct consequence of the origin supported condition for the filtration).

Any such filtration gives a way to express the $\mathcal{A}_e$-codimension as an infinite sum, or as a "degree r aproximation" plus a tail, that is:

(1) $\operatorname{codim}_{\mathcal{A}_e}(f) = d_0 + d_1 + \dots$
(2) $\operatorname{codim}_{\mathcal{A}_e}(f) = d_0 + d_1 + \cdots + d_r + \dim_{\mathbb{C}} \frac{\psi^{-1}(T\mathcal{A}_e(f)) + M_{r+1}}{\psi^{-1}(T\mathcal{A}_e(f))}$

Both expressions are helpful in different situations and allow us to compute $\operatorname{codim}_{\mathcal{A}_e}(f)$ by a series of aproximations of $\psi^{-1}(T\mathcal{A}_e(f))$, defined as follows.

Definition 1.8. Given a map f as above and a filtration $M_\bullet$ of $\bigoplus^p \bigoplus^m \mathcal{O}_n$. An *approximating sequence* for $T\mathcal{A}_e(f)$ (subject to $M_\bullet$) is a sequence $A_\bullet$ of $\mathcal{O}_n$-submodules $A_r \leq \bigoplus^p \bigoplus^m \mathcal{O}_n$, called *approximations*, such that

$$A_r \equiv \psi^{-1}(T\mathcal{A}_e(f)) \mod M_{r+1}$$

or, equivalently,

$$\psi(A_r) \equiv T\mathcal{A}_e(f) \mod \psi(M_{r+1}).$$

Then, clearly we have that

$$d_r = \dim_{\mathbb{C}} \frac{A_r + M_r}{A_r + M_{r+1}}.$$

1.4. **The computation method.** The notions of the previous sections gives two strategies to compute $\operatorname{codim}_{\mathcal{A}_e}(f)$ by computing the successive approximations A_r:

We start computing A_r and d_r for $r = 0, 1, 2, \dots$ and, for each r,

(1) If $d_r = 0$, then find the biggest number L such that

$$d_{r-L} = \cdots = d_{r-1} = d_r = 0.$$

If L satisfies something which can be decided directly from the expression of f (This is Du Plessis' criterion, which we need to adapt slightly to our setup), then all d_j are zero for $j \geq r$. In that case,

$$\operatorname{codim}_{\mathcal{A}_e}(f) = d_0 + \cdots + d_{r-L-1}.$$

(2) If $\psi(A_r) = T\mathcal{A}_e(f)$, then

$$\operatorname{codim}_{\mathcal{A}_e}(f) = d_0 + d_1 + \cdots + d_r + \dim_{\mathbb{C}} \frac{A_r + M_{r+1}}{A_r}$$

Remark 1.9. On one hand, strategy (1) finishes if and only if $T\mathcal{A}_e(f) < \infty$. In particular, it is unable to decide whether of not f is $\mathcal{A}$-finite. On the other hand, we are only able to use strategy (2) if at some point our approximation is exact. But, if we do, we are able to decide whether or not f is $\mathcal{A}$-finite in finite time, since this happens if and only if $\dim_{\mathbb{C}} \frac{A_r + M_{r+1}}{A_r} < \infty$

Remark 1.10. Alternatively (but probably not equivalently in terms of computing efficiency) we may define

$$D_r = \dim_{\mathbb{C}} \frac{\bigoplus^p \bigoplus^m \mathcal{O}_n}{\psi^{-1}(T\mathcal{A}_e(f)) + M_{r+1}} = \dim_{\mathbb{C}} \frac{\bigoplus^p \bigoplus^m \mathcal{O}_n}{A_r + M_{r+1}}.$$

Then we have an increasing sequence $D_0 \leq D_1 \leq \dots$ such that

$$\operatorname{codim}_{\mathcal{A}_e}(f) = \lim_{r \to \infty} D_r$$

and obviously $D_r = d_0 + \dots d_r$, The two strategies above now look as follows: We start computing A_r and D_r for $r = 0, 1, 2, \cdots$, and for each r

(1) If $D_r = D_{r-1}$, then find the biggest number L such that

$$D_{r-L} = \cdots = D_{r-1} = D_r = 0.$$

There is a criterion by Du Plessis, computable in terms of A_r, which ensures that, if L satisfies something (We need to make this precise), then all remaining D_j for $j \geq r$ are equal to D_r. Consequently,

$$\operatorname{codim}_{\mathcal{A}_e}(f) = D_r.$$

(2) If $\psi(A_r) = T\mathcal{A}_e(f)$, then

$$\operatorname{codim}_{\mathcal{A}_e}(f) = D_r$$

2. The algorithm

Assume that, given a map-germ f, for every integer r, we have

(1) The module M_r of a filtration $M_\bullet$ of $\bigoplus^p \bigoplus^m \mathcal{O}_n$.

(2) A procedure to compute, for any vector $v \in \theta(f)$ and every integer $r \geq 0$, a matrix H such that

$$\psi(H) \equiv v \mod \psi(M_{r+1}).$$

(Although not strictly necessary, it isconvenient to ask for this procedure to return $H = 0$ if $v \in \psi(M_{r+1})$, or at least if $v = 0$).

Then, the computational method of Section 1.4 can be implemented as follows:

INPUT:

- The map f,
- The generators $g_1, \ldots, g_m$ of $\overline{f}_* \mathcal{O}_X$.
- (Optional) a degree bound K.

OUTPUT:

- An integer a, which may be of the following forms:
  - $a \geq 0$, in which case $\operatorname{codim}_{\mathcal{A}_e}(f) = a$
  - $a = -1$, in which case $\operatorname{codim}_{\mathcal{A}_e}(f) = \infty$
  - $a = -2$, in which case the degree bound K has been surpassed during calculations (in a sense to be made precise later)

STEPS:

(1) SET
- $a = -2$
- $r = 0$
- $M = \bigoplus^p \bigoplus^m \mathcal{O}_n$
- SET $v_1, \ldots, v_q$ = the collection of generators of $T\mathcal{A}_e(f)$ as an $\mathcal{O}_n$-module (see Proposition 1.2)
- $V_1, \ldots, V_q = 0$

(2) IF r IS TOO BIG FOR K (To be made precise)
- RETURN a

(3) ELSE:

(a) FOR $i = 1, \ldots, q$
- FIND A MATRIX R_i, SUCH THAT

$$\psi(R_i) \equiv v_i \mod \psi(M_{r+1}).$$

(See Remark 2.1)
- $V_i = V_i + R_i$
- $v_i = v_i - \psi(R_i)$

(b) $d_r = \dim_{\mathbb{C}} \dfrac{\langle V_1, \ldots, V_q \rangle + M_r}{\langle V_1, \ldots, V_q \rangle + M_{r+1}}$

(c) IF $v_1 = \cdots = v_q = 0$, THEN
- $T = \dim_{\mathbb{C}} \dfrac{\langle V_1, \ldots, V_q \rangle + M_r}{\langle V_1, \ldots, V_q \rangle + M_{r+1}}$
- IF $T = -1$
  - RETURN T
- ELSE,
  - RETURN $d_0 + \cdots + d_r + T$

(d) ELSE

- IF $d_r = 0$ AND WE HAVE ENOUGH CONSECUTIVE $d_{r-L} = \cdots = d_r = 0$ (To be made precise),
  - RETURN $d_0 + \cdots + d_{r-L-1}$.
- ELSE,
  - $r = r + 1$
  - GO TO STEP 2

Remark 2.1. Step 3a involves a huge amount of calculations. Remarks 1.3 and 1.4 should be taken into account in order to reduce it. Without being very precise, in principle, it should be enough to compute at most the consecutive approximations of certain set of monomials, or of the products, $x_i g_j$ or something like that

Remark 2.2. If we replace d_r by D_r as in Remark 1.10 and choose M_r adequately, the calculations of D_r required are calculations in a free module over a quotient ring, which could speed up calculations.

3. Filtration choices and their associated approximations

The method explained above requires to

(1) Chose a filtration $M_\bullet$ as in Definition 1.5
(2) Computing aproximations $H \in M_r$ of residues $v \in \psi(M_r)$ modulo the chosen filtration:

We must take into account that at this point we do not know how to compute approximations modulo an arbitrary submodule $\psi(M_r)$, but only approximations of the form

$$\psi(H) \equiv v \mod \bigoplus^p \mathfrak{m}_X^N.$$

Theoretically, this is not a problem, because the origin-supported condition together with the finiteness of $\overline{f}$ (see Remark 1.6) ensure that, for every r, there is some exponent N such that

$$\bigoplus^p \mathfrak{m}_X^N \subseteq M_r$$

and clearly an approximation modulo $\bigoplus^p \mathfrak{m}_X^N$ is a valid approximation modulo M_r as well.

Remark 3.1. Notice that we do not need to compute N before proceeding to compute the approximations. We can fix a tentative N, compute the approximation modulo $\bigoplus^p \mathfrak{m}_X^N$. Then compute its residue modulo $\psi(M_r)$. If the residue is zero, the desired approximation has been obtained. Otherwise, the residue of this approximation is approximated again with N replaced by $N + 1$.

It is however likely that the choice of the filtration affects the efficiency of the algorithm. The idea of this section is to collect the different choices that we want to experiment with.

3.1. **The target-degree filtration.** For each r, take

$$M_r = \bigoplus^{p} \bigoplus^{m} \mathfrak{m}_n^r$$

For this filtration, if we let

$$\operatorname{ord}(\overline{f}) = \max\{\operatorname{ord}(f_1), \ldots, \operatorname{ord}(f_n)\} \qquad \rho = \max\{\operatorname{ord}(g_1), \ldots, \operatorname{ord}(g_n)\},$$

For each r, our tentative initial N is

$$N = r \cdot \operatorname{ord}(\overline{f}) + \rho$$

3.2. **The source-degree filtration.** For each r, take

$$\mathfrak{n}_{k,r} = \{X^\alpha \mid \sum_{j=1}^{n} \alpha_j \cdot \operatorname{ord}(f_j) \operatorname{ord}(g_k) \geq r\}$$

The tentative initial N is

$$N = r$$

4. Procedures

(1) GeneratingSetOverC:
INPUT: list of polynomials $H = h_1, \ldots, h_n$
OUTPUT:
- basis $Q = q_1, \ldots, q_r$ of $\operatorname{Span}_{\mathbb{C}}\{h_1, \ldots, h_n\}$.
- $n \times r$ matrix M with entries in $\mathbb{C}$ such that

$$H = MQ$$

Should consist just on calling

$$coef(ideal(h1, \ldots, h_n), x_1 \cdot \cdots \cdot x_n))$$

and applying Gauss reduction to the resulting matrix while keeping track of the steps to undo the process.

Remark: Ideally, we would provide a generating set over $\mathcal{O}_n$, and a matrix M with entries in $\mathcal{O}_n$, but producing this may be too costly.

(2) ListEntries:
INPUT: list of vectors $V = v_1, \ldots, v_n$ of size p
OUTPUT:
- list of polinomials $h_1, \ldots, h_m$ consisting of the entries of v_i (in order of appearance)
- the integer p

(3) AproxModuloM:
INPUT:
- polynomial h
- module M

OUTPUT:
- vector $H = H_1, \ldots, H_m$ such that

$$h \equiv \varphi(H) \mod M$$

5. To do list

(1) Minimizing ammount of approximations: Much of the calculations are repeated, simply because they appear at different entries of vectors. It seems that, for every graded piece M_r/M_{r+1} we should be pin down a set of monomials that need to be approximated, and use these to write down all the required vectors. Also, we may be able to exploit the expressions of the monomials in lower degree to speed up the current degree calculations.

Idea: replace $v_1, \ldots, v_q$ by a basis of the $\mathbb{C}$-vector space spanned by all entries of $v_1, \ldots, v_q$.

(2) Replacement of generators: Figure out whether a generator v can be replaced by a combination $\tilde{v}$ of the existing generators with target coefficients, in a way such that
   - The submodule generated is the same using v or $\tilde{v}$
   - The approximation of $\tilde{v}$ is exact