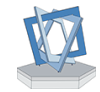
Algebraic, geometric and topological properties of complex projective plane curves

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- Complex algebraic curve: Riemann surface with singularities.
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  - Topological type of singularities:
    - Algebraic meaning (equisingular type)
    - Link in $\mathbb{S}^3$
    - Smooth point, $y = 0$, unknot
    - Node: $y^2 - x^2 = 0$, Hopf link
    - Ordinary cusp: $y^2 - x^3 = 0$, trefoil knot
    - $y^p - x^q = 0$, (p, q) torus link



Local and global properties

Local

- Singularity defined by $\{f(x,y)=0\}$, $f \in \mathbb{C}\{x,y\}$ reduced:
  - Milnor number: $\mu = \dim_{\mathbb{C}} \mathbb{C}\{x,y\}/\operatorname{Jac} f$
  - Milnor fiber: $\Sigma_t : f(x,y) = t$ in a small ball, $\chi(\Sigma_t) = 1-\mu$



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- $2\delta = \mu + r - 1$, $r = \#$ of irreducible factors of f (only in characteristic 0).
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- $d = 1, 2$: rational curves.
- $d = 3$: elliptic if smooth, rational if singular.
- $d = 6$, 6 ordinary cusps $g = 4$.



Embedded topology

- $\mathcal{C}$ irreducible of degree d: $H_1(\mathbb{P}^2 \smallsetminus \mathcal{C}; \mathbb{Z}) \cong \mathbb{Z}/d$.
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- $\mathcal{C}$ degree 6 with six cusps NON on a conic: $\nexists \pi_1(\mathbb{P}^2 \setminus \mathcal{C}) \twoheadrightarrow \mathfrak{S}_3$, actually $\cong \mathbb{Z}/2 \times \mathbb{Z}/3$



Zariski pairs

Definition

The combinatorics of $\mathcal{C} \subset \mathbb{P}^2$ is the topological type of $(\mathcal{T}, \mathcal{C})$, $\mathcal{T}$ closed regular neighbourhood of $\mathcal{C}$.

- $\mathcal{C}$ arrangement of lines: combinatorics $\equiv$ intersection pattern
- $\mathcal{C}$ irreducible: combinatorics $\equiv$ degree and topological type of singularities



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$\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{P}^2$ Zariski pair if same combinatorics $\nexists (\mathbb{P}^2, \mathcal{C}_1) \xrightarrow{\text{homeo}} (\mathbb{P}^2, \mathcal{C}_2)$

- First example: Zariski sextics



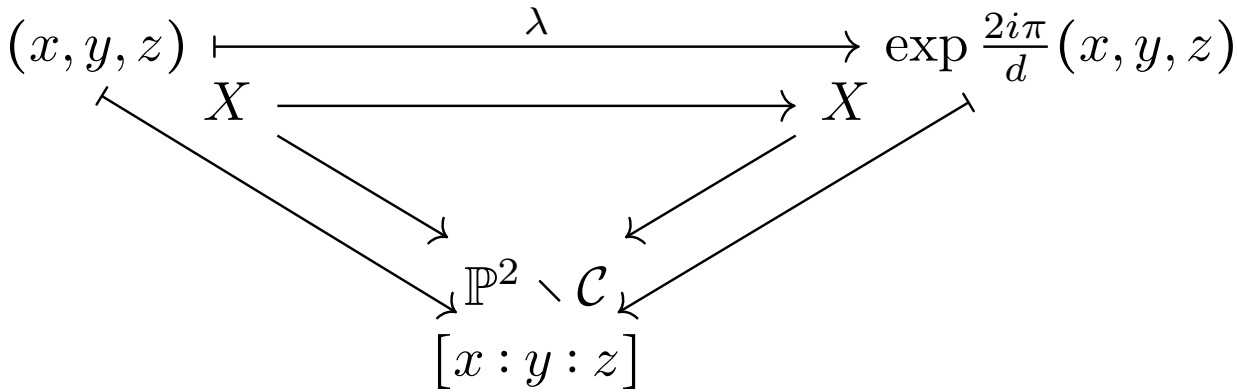
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$\Delta(t)$ Alexander polynomial: characteristic polynomial of

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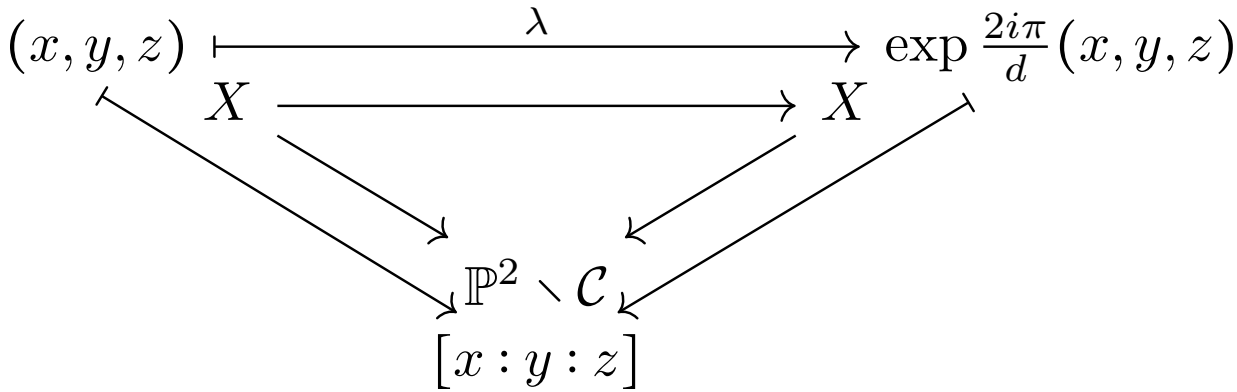
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- $\mathcal{C}$ Zariski sextic six points on a conic: $\Delta(t) = t^2 - t + 1$
- $\mathcal{C}$ Zariski sextic six points NOT on a conic: $\Delta(t) = 1$



Characteristics of Zariski pairs.

- $\exists$ Zariski pairs not Alexander Zariski (arrangements of cubics and conics):
  - As Zariski pairs distinguished by π_1
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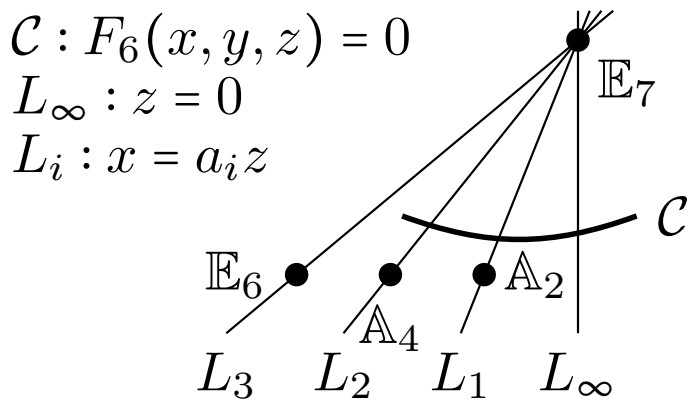
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- $\exists$ Zariski pairs with homeomorphic complements
- $\exists$ arithmetic Zariski pairs

Definition

$\mathbb{K}$ number field, $\mathcal{C} \subset \mathbb{K}\mathbb{P}^2$ curve: $\{\mathcal{C} \otimes_\varphi \mathbb{C} \mid \varphi : \mathbb{K} \hookrightarrow \mathbb{C}\}$ arithmetic Zariski tuple if pairwise Zariski pairs



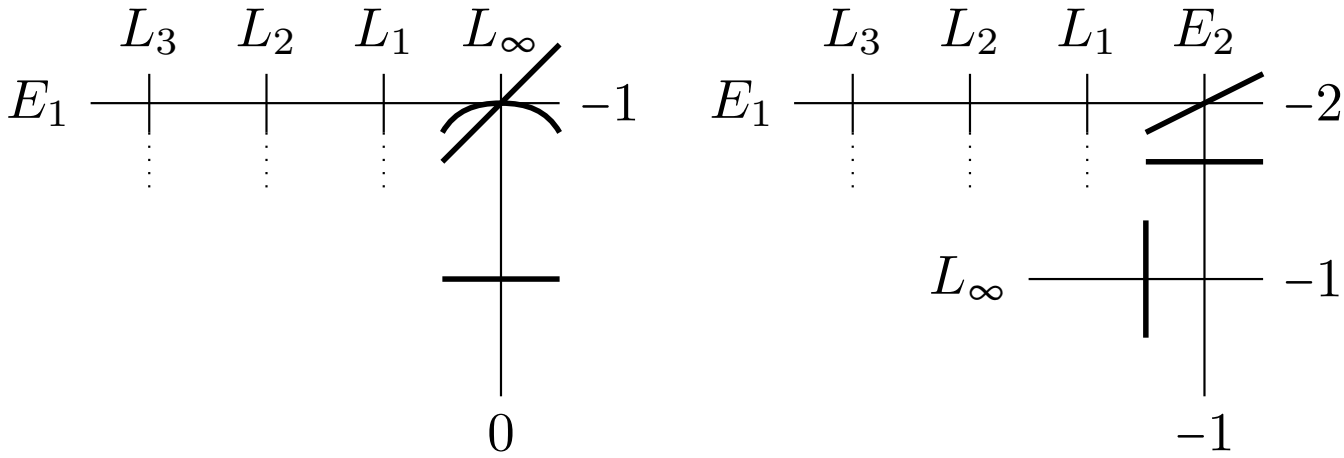
Complement-homeomorphic arithmetic Zariski pair



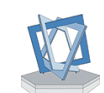
- $F_6 \in \mathbb{Q}[\sqrt{5}][x,y,z]$, $f(x,y) := F_6(x,y,1)$ y-monic, $\deg_y f = 3$
- $\varphi \in \mathrm{Gal}\,\mathbb{Q}[\sqrt{5}]/\mathbb{Q}$, $\varphi(\sqrt{5}) = -\sqrt{5}$
- $\exists a,b,c,d,e \in \mathbb{Q}[\sqrt{5}]^*$ s.t. $f^\varphi(x, -y + ax + bx^2) = cf(x, dx + ey)$.
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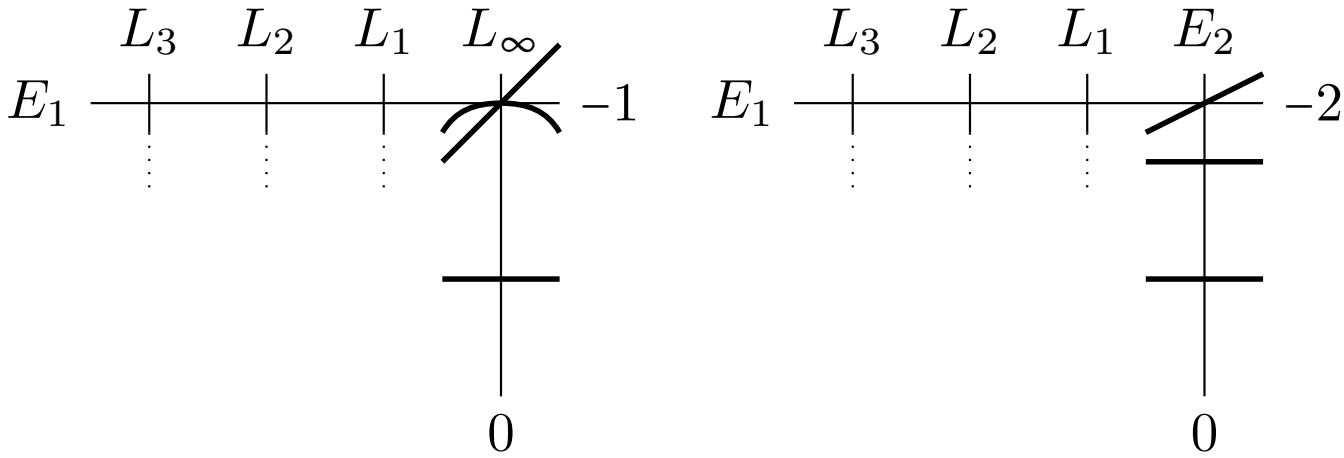
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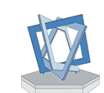
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Triangular combinatorics

Combinatorics $\mathscr{C}$ of degree d

- $\Sigma^{\mathscr{C}} := \{\text{curves satisfying } \mathscr{C}\}$ quasiprojective variety
- $\mathscr{M}^{\mathscr{C}} := \Sigma^{\mathscr{C}} / \mathrm{PGL}(3;\mathbb{C})$



Triangular combinatorics

$\mathscr{M}_{d,1} = \mathscr{M}_1^{d+3} := \mathscr{M}^{\mathscr{C}_1}$, $d > 1$

$\mathscr{C}_1$: irreducible components $\mathcal{C}_d, L_1^1, L_1^2, L_1^3$ (subscript is degree)

- $\mathcal{C}_d$ smooth
- $\#\mathcal{C}_d \cap L_1^i = 1$
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- M_1^i joining two singular points
- $\mathscr{C}_1 \leftrightarrow \mathscr{C}_2$ via standard Cremona transformation



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Moduli spaces

Theorem

$d > 2,\ \zeta = \exp\frac{2i\pi}{d},\ \mathscr{M}_{d,j} = \bigcup_{k=0}^{\lfloor \frac{d}{2} \rfloor} \mathscr{M}_{d,j}^{\zeta^k}$

- $\mathscr{M}_{d,j}^{\zeta^k} \neq \varnothing$ *and connected.*
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- $\left\{\mathcal{C}_{d,j,k} \mid 0 \leq k \leq \left\lfloor \frac{d}{2} \right\rfloor\right\}$ *is a Zariski tuple for:*
  - $d = 3$
  - $j = 1,2$ *and* $d > 3$
- *They contain arithmetic Zariski tuples for* $d > 4.$



Moduli spaces

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Open problem

Is $\left\{\mathcal{C}_{d,3,k} \mid 0 \leq k \leq \left\lfloor \frac{d}{2} \right\rfloor\right\}$ a Zariski tuple for $d > 3$?



Construction of the curves

Proposition

For $\mathcal{C}_d + L_1^1 + L_1^2 + L_1^3 \in \Sigma^{\mathscr{C}_1}$, *there is an automorphism* $\Phi \in \mathrm{PGL}(3;\mathbb{C})$ *such that*

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$\mathcal{C}_d = \{G_d(x,y,z) = 0\}$ where

- $G_d(0,y,z) = (y+z)^d$, $G_d(x,0,z) = (x+z)^d$



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- For $\Sigma^{\mathscr{C}_2}$ apply $[x:y:z] \mapsto [yz:xz:xy]$
- For $\Sigma^{\mathscr{C}_3}$ forget the lines.



Zariski tuples

Sketch of the proof.

- $V = \{[x : y : z : t] \in \mathbb{P}^3 \mid t^d - G_d(x, y, z) = 0\}$

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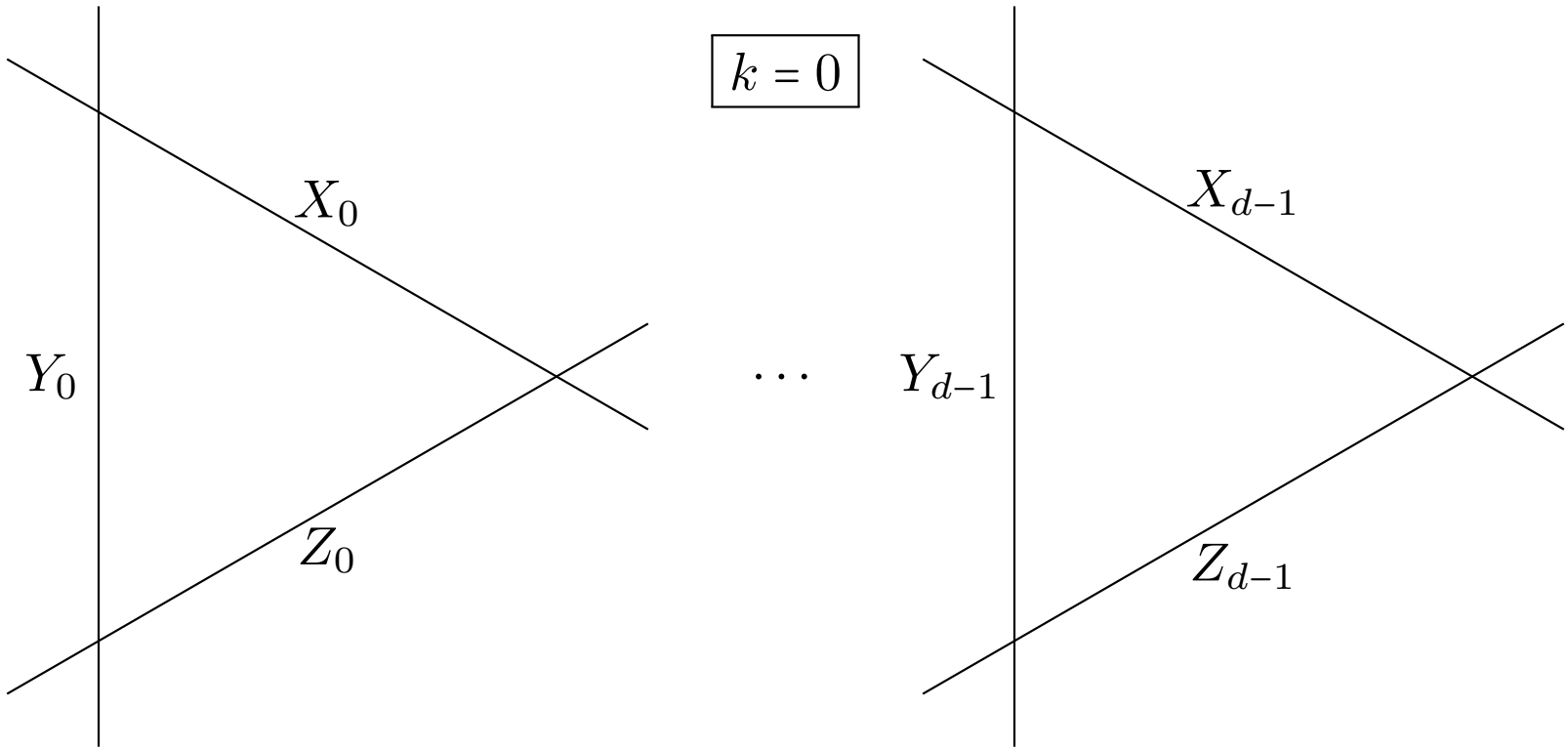
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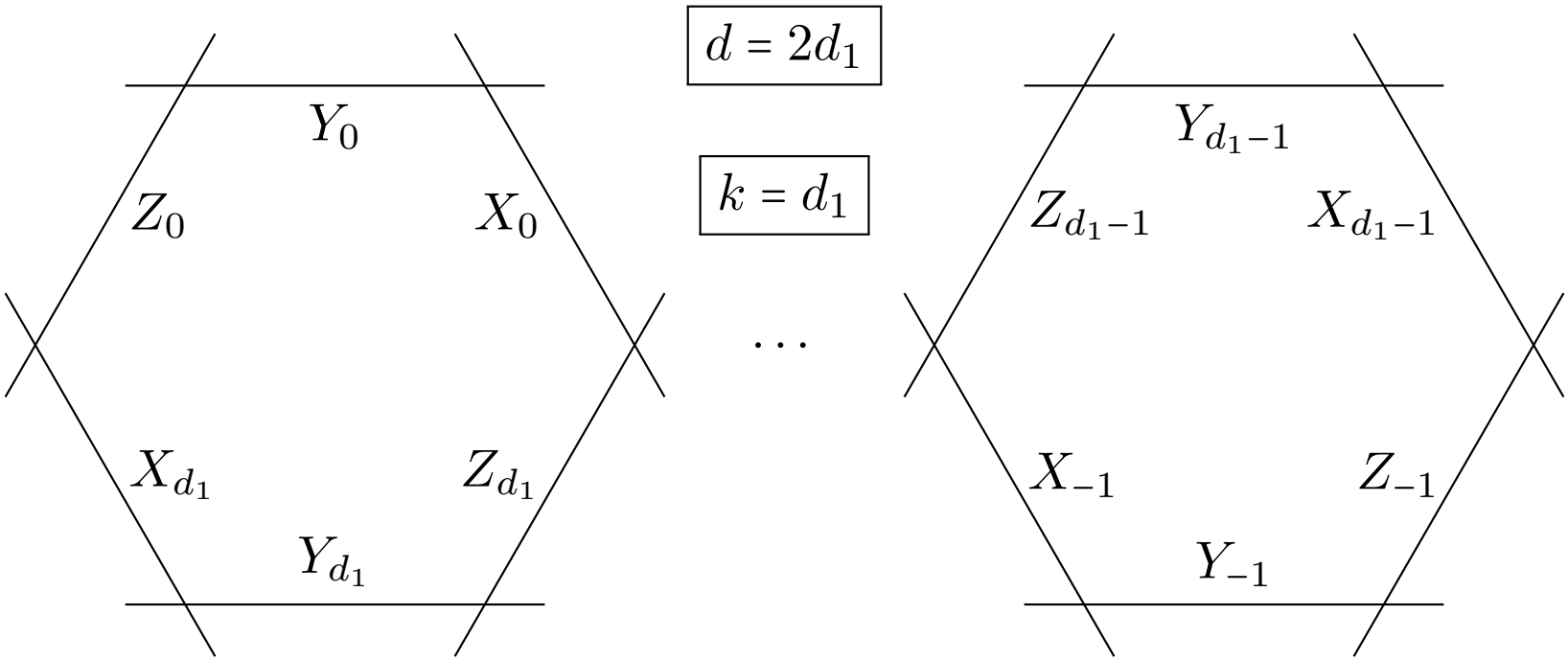
Apply covering arguments □



Preimages of the lines

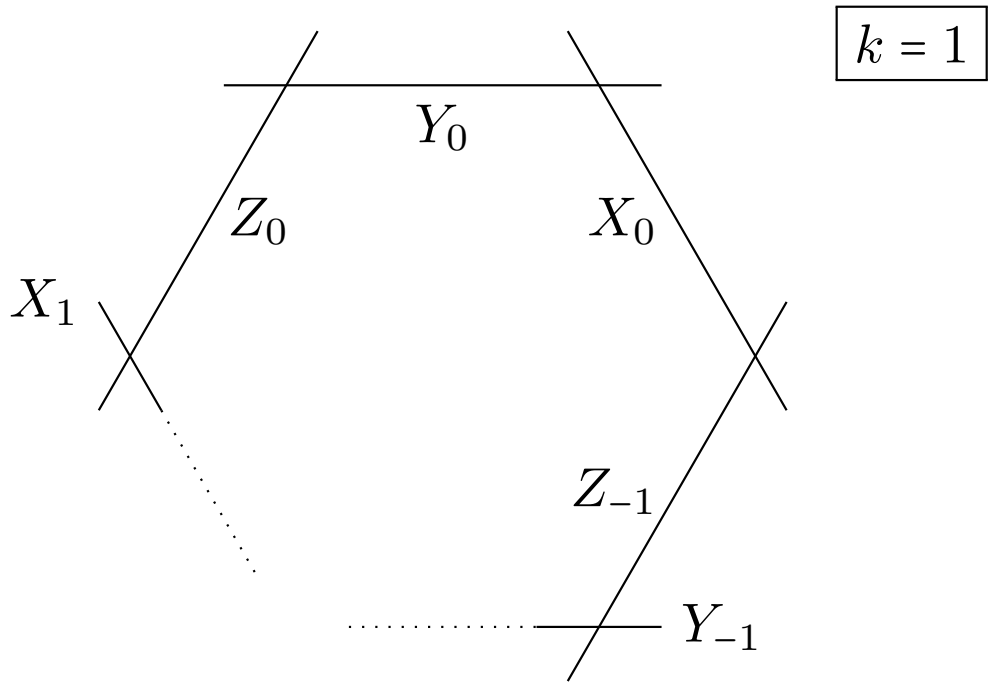


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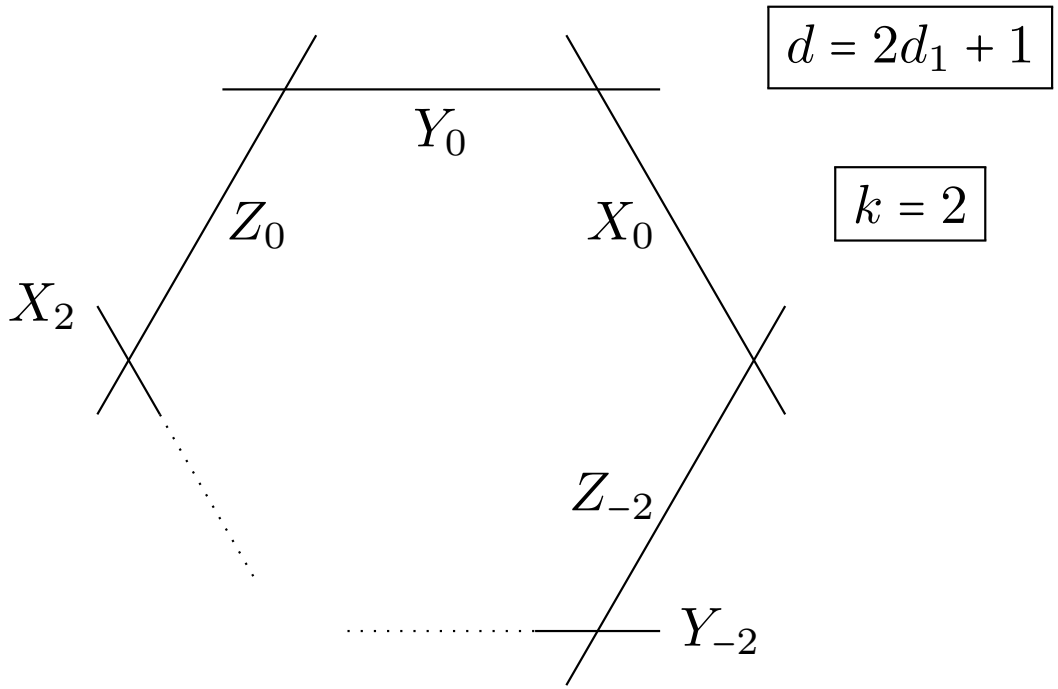




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Thank you for your attention!

