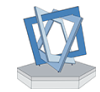
How can one check if a tuple of curves is a Zariski tuple?

Enrique ARTAL BARTOLO

Departamento de Matemáticas
Facultad de Ciencias
Instituto Universitario de Matemáticas y sus Aplicaciones
Universidad de Zaragoza

Iberoamerican Web(m)inar of Young Researchers in Singularity Theory and related topics
October 6th 2021

Funded by MCIN/AEI/10.13039/501100011033 (PID2020-114750GB-C31) and Gobierno de Aragón (E22_20R: "Álgebra y Geometría").



Irreducible Zariski tuple candidates

_, J.I. Cogolludo, and J. Martín. "Triangular Curves and Cyclotomic Zariski Tuples." **Collect. Math. 71** (3) 2020, 427–41



Irreducible Zariski tuple candidates

_, J.I. Cogolludo, and J. Martín. "Triangular Curves and Cyclotomic Zariski Tuples." **Collect. Math. 71** (3) 2020, 427–41

- $\mathscr{M}^{2d}_{3(d,d+1)}$: $\mathcal{C}_{2d} \subset \mathbb{P}^2$, $\#\operatorname{Sing}(\mathcal{C}_{2d}) = 3$, singularities $\overset{\text{top}}{\sim} \{u^d = v^{d+1}\}$.



Irreducible Zariski tuple candidates

_, J.I. Cogolludo, and J. Martín. "Triangular Curves and Cyclotomic Zariski Tuples." **Collect. Math. 71** (3) 2020, 427–41

- $\mathscr{M}^{2d}_{3(d,d+1)}$: $\mathcal{C}_{2d} \subset \mathbb{P}^2$, $\#\operatorname{Sing}(\mathcal{C}_{2d}) = 3$, singularities $\overset{\text{top}}{\sim} \{u^d = v^{d+1}\}$.
- Are two such curves topologically equivalent in $\mathbb{P}^2$?



Irreducible Zariski tuple candidates

_, J.I. Cogolludo, and J. Martín. "Triangular Curves and Cyclotomic Zariski Tuples." **Collect. Math. 71** (3) 2020, 427–41

- $\mathscr{M}^{2d}_{3(d,d+1)}$: $\mathcal{C}_{2d} \subset \mathbb{P}^2$, $\#\operatorname{Sing}(\mathcal{C}_{2d}) = 3$, singularities $\overset{\text{top}}{\sim} \{u^d = v^{d+1}\}$.
- Are two such curves topologically equivalent in $\mathbb{P}^2$?
- $d = 2$: tricuspidal quartic, projectively equivalent.



Irreducible Zariski tuple candidates

_, J.I. Cogolludo, and J. Martín. "Triangular Curves and Cyclotomic Zariski Tuples." **Collect. Math. 71** (3) 2020, 427–41

- $\mathscr{M}^{2d}_{3(d,d+1)}$: $\mathcal{C}_{2d} \subset \mathbb{P}^2$, $\#\operatorname{Sing}(\mathcal{C}_{2d}) = 3$, singularities $\overset{\text{top}}{\sim} \{u^d = v^{d+1}\}$.
- Are two such curves topologically equivalent in $\mathbb{P}^2$?
- $d = 2$: tricuspidal quartic, projectively equivalent.
- $d = 3$: sextic with 3 $\mathbb{E}_6$, topology depends on the existence of a conic tangent to the singularities.



Irreducible Zariski tuple candidates

_, J.I. Cogolludo, and J. Martín. "Triangular Curves and Cyclotomic Zariski Tuples." **Collect. Math. 71** (3) 2020, 427–41

- $\mathscr{M}^{2d}_{3(d,d+1)}$: $\mathcal{C}_{2d} \subset \mathbb{P}^2$, $\#\operatorname{Sing}(\mathcal{C}_{2d}) = 3$, singularities $\overset{\text{top}}{\sim} \{u^d = v^{d+1}\}$.
- Are two such curves topologically equivalent in $\mathbb{P}^2$?
- $d = 2$: tricuspidal quartic, projectively equivalent.
- $d = 3$: sextic with 3 $\mathbb{E}_6$, topology depends on the existence of a conic tangent to the singularities.
  - Distinct fundamental groups



Irreducible Zariski tuple candidates

_, J.I. Cogolludo, and J. Martín. "Triangular Curves and Cyclotomic Zariski Tuples." **Collect. Math. 71** (3) 2020, 427–41

- $\mathscr{M}^{2d}_{3(d,d+1)}$: $\mathcal{C}_{2d} \subset \mathbb{P}^2$, $\#\operatorname{Sing}(\mathcal{C}_{2d}) = 3$, singularities $\overset{\text{top}}{\sim} \{u^d = v^{d+1}\}$.
- Are two such curves topologically equivalent in $\mathbb{P}^2$?
- $d = 2$: tricuspidal quartic, projectively equivalent.
- $d = 3$: sextic with 3 $\mathbb{E}_6$, topology depends on the existence of a conic tangent to the singularities.
  - Distinct fundamental groups
  - Distinct Alexander polynomials



Irreducible Zariski tuple candidates

_, J.I. Cogolludo, and J. Martín. "Triangular Curves and Cyclotomic Zariski Tuples." **Collect. Math. 71** (3) 2020, 427–41

- $\mathscr{M}^{2d}_{3(d,d+1)}$: $\mathcal{C}_{2d} \subset \mathbb{P}^2$, $\#\operatorname{Sing}(\mathcal{C}_{2d}) = 3$, singularities $\overset{\text{top}}{\sim} \{u^d = v^{d+1}\}$.
- Are two such curves topologically equivalent in $\mathbb{P}^2$?
- $d = 2$: tricuspidal quartic, projectively equivalent.
- $d = 3$: sextic with 3 $\mathbb{E}_6$, topology depends on the existence of a conic tangent to the singularities.
  - Distinct fundamental groups
  - Distinct Alexander polynomials
- $d = 4$: a general member (e.g, $a = 1, b = -1, c = 0$) of

$$y^4(x+z)^4 + x^4(y+z)^4 + z^4(x+\lambda y)^4 - (y^4z^4 + x^4z^4 + x^4y^4)$$
$$+4x^2y^2z^3(ax+by) + cx^3y^3z^2 = 0, \qquad a, b, c \in \mathbb{C},$$
$$\lambda^4 = 1 \qquad \lambda = 1, -1, i.$$



Topology and algebra of algebraic curves

Embedded homeomorphism types

- $\mathcal{C}^1, \mathcal{C}^2 \subset \mathbb{P}^2$, $\mathcal{C}^j = \mathcal{C}^j_1 \cup \cdots \cup \mathcal{C}^j_{r_j}$ irreducible decompositions.



Topology and algebra of algebraic curves

Embedded homeomorphism types

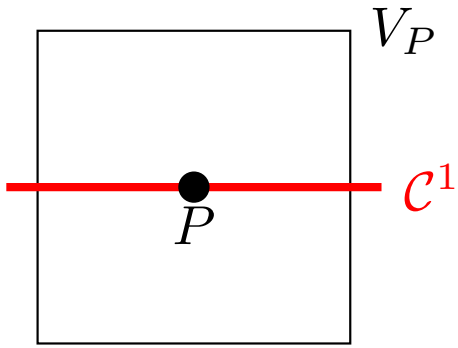
- $\mathcal{C}^1, \mathcal{C}^2 \subset \mathbb{P}^2$, $\mathcal{C}^j = \mathcal{C}^j_1 \cup \cdots \cup \mathcal{C}^j_{r_j}$ irreducible decompositions.
- $\exists \Phi : (\mathbb{P}^2, \mathcal{C}^1) \to (\mathbb{P}^2, \mathcal{C}^2)$ homeomorphism?

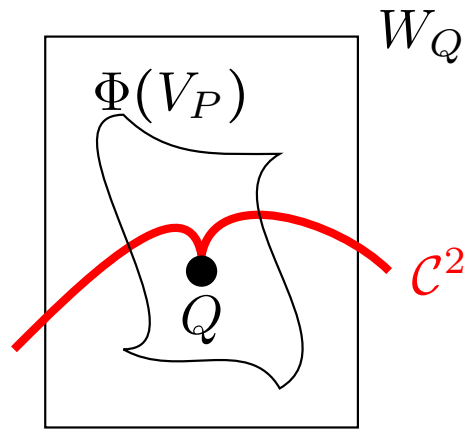


Topology and algebra of algebraic curves

Embedded homeomorphism types

- $\mathcal{C}^1, \mathcal{C}^2 \subset \mathbb{P}^2$, $\mathcal{C}^j = \mathcal{C}^j_1 \cup \cdots \cup \mathcal{C}^j_{r_j}$ irreducible decompositions.
- $\exists \Phi : (\mathbb{P}^2, \mathcal{C}^1) \to (\mathbb{P}^2, \mathcal{C}^2)$ homeomorphism?
- $P \in \mathcal{C}^1$ smooth implies $Q := \Phi(P) \in \mathcal{C}^2$ smooth?







Topology and algebra of algebraic curves

Embedded homeomorphism types

- $\mathcal{C}^1, \mathcal{C}^2 \subset \mathbb{P}^2$, $\mathcal{C}^j = \mathcal{C}^j_1 \cup \cdots \cup \mathcal{C}^j_{r_j}$ irreducible decompositions.
- $\exists \Phi : (\mathbb{P}^2, \mathcal{C}^1) \to (\mathbb{P}^2, \mathcal{C}^2)$ homeomorphism?
- $P \in \mathcal{C}^1$ smooth implies $Q := \Phi(P) \in \mathcal{C}^2$ smooth?

Local fundamental group

- $\pi_1^{\mathrm{loc}}(\mathbb{P}^2, \mathcal{C}^1, P) := \lim_{P \in V \text{ nghbd}} \pi_1(V \setminus \mathcal{C}^1)$



Topology and algebra of algebraic curves

Embedded homeomorphism types

- $\mathcal{C}^1, \mathcal{C}^2 \subset \mathbb{P}^2$, $\mathcal{C}^j = \mathcal{C}^j_1 \cup \cdots \cup \mathcal{C}^j_{r_j}$ irreducible decompositions.
- $\exists \Phi : (\mathbb{P}^2, \mathcal{C}^1) \to (\mathbb{P}^2, \mathcal{C}^2)$ homeomorphism?
- $P \in \mathcal{C}^1$ smooth implies $Q := \Phi(P) \in \mathcal{C}^2$ smooth?

Local fundamental group

- $\pi_1^{\mathrm{loc}}(\mathbb{P}^2, \mathcal{C}^1, P) := \lim_{P \in V \text{ nghbd}} \pi_1(V \setminus \mathcal{C}^1)$
- $P \in \mathcal{C}^1$ smooth $\iff \pi_1^{\mathrm{loc}}(\mathbb{P}^2, \mathcal{C}^1, P) \cong \mathbb{Z}$.



Topology and algebra of algebraic curves

Embedded homeomorphism types

- $\mathcal{C}^1, \mathcal{C}^2 \subset \mathbb{P}^2$, $\mathcal{C}^j = \mathcal{C}^j_1 \cup \cdots \cup \mathcal{C}^j_{r_j}$ irreducible decompositions.
- $\exists \Phi : (\mathbb{P}^2, \mathcal{C}^1) \to (\mathbb{P}^2, \mathcal{C}^2)$ homeomorphism?
- $P \in \mathcal{C}^1$ smooth implies $Q := \Phi(P) \in \mathcal{C}^2$ smooth



Topology and algebra of algebraic curves

Embedded homeomorphism types

- $\mathcal{C}^1, \mathcal{C}^2 \subset \mathbb{P}^2$, $\mathcal{C}^j = \mathcal{C}^j_1 \cup \cdots \cup \mathcal{C}^j_{r_j}$ irreducible decompositions.
- $\exists \Phi : (\mathbb{P}^2, \mathcal{C}^1) \to (\mathbb{P}^2, \mathcal{C}^2)$ homeomorphism?
- $P \in \mathcal{C}^1$ smooth implies $Q := \Phi(P) \in \mathcal{C}^2$ smooth
- $r_1 = r_2 = r$. After reordering $\Phi(\mathcal{C}^1_i) = \mathcal{C}^2_i$.



Topology and algebra of algebraic curves

Embedded homeomorphism types

- $\mathcal{C}^1, \mathcal{C}^2 \subset \mathbb{P}^2$, $\mathcal{C}^j = \mathcal{C}^j_1 \cup \cdots \cup \mathcal{C}^j_{r_j}$ irreducible decompositions.
- $\exists \Phi : (\mathbb{P}^2, \mathcal{C}^1) \to (\mathbb{P}^2, \mathcal{C}^2)$ homeomorphism?
- $P \in \mathcal{C}^1$ smooth implies $Q := \Phi(P) \in \mathcal{C}^2$ smooth
- $r_1 = r_2 = r$. After reordering $\Phi(\mathcal{C}^1_i) = \mathcal{C}^2_i$.
- $H_1(\mathbb{P}^2 \setminus \mathcal{C}^j_i; \mathbb{Z}) = \mathbb{Z}/\deg \mathcal{C}^j_i \Longrightarrow \deg \mathcal{C}^1_i = \deg \mathcal{C}^2_i$.



Topology and algebra of algebraic curves

Embedded homeomorphism types

- $\mathcal{C}^1, \mathcal{C}^2 \subset \mathbb{P}^2$, $\mathcal{C}^j = \mathcal{C}^j_1 \cup \cdots \cup \mathcal{C}^j_{r_j}$ irreducible decompositions.
- $\exists \Phi : (\mathbb{P}^2, \mathcal{C}^1) \to (\mathbb{P}^2, \mathcal{C}^2)$ homeomorphism?
- $P \in \mathcal{C}^1$ smooth implies $Q := \Phi(P) \in \mathcal{C}^2$ smooth
- $r_1 = r_2 = r$. After reordering $\Phi(\mathcal{C}^1_i) = \mathcal{C}^2_i$.
- $H_1(\mathbb{P}^2 \setminus \mathcal{C}^j_i; \mathbb{Z}) = \mathbb{Z}/\deg \mathcal{C}^j_i \Longrightarrow \deg \mathcal{C}^1_i = \deg \mathcal{C}^2_i$.

Local fundamental group

- $\pi_1^{\mathrm{loc}}(\mathbb{P}^2, \mathcal{C}^1, P) := \lim_{P \in V \text{ nghbd}} \pi_1(V \setminus \mathcal{C}^1)$
- $\pi_1^{\mathrm{loc}}(\mathbb{P}^2, \mathcal{C}^1, P)$ determines local topological type of $(\mathcal{C}^1, P)$ irreducible (Alexander polynomial, too).



Topology and algebra of algebraic curves

Embedded homeomorphism types

- $\mathcal{C}^1, \mathcal{C}^2 \subset \mathbb{P}^2$, $\mathcal{C}^j = \mathcal{C}^j_1 \cup \cdots \cup \mathcal{C}^j_{r_j}$ irreducible decompositions.
- $\exists \Phi : (\mathbb{P}^2, \mathcal{C}^1) \to (\mathbb{P}^2, \mathcal{C}^2)$ homeomorphism?
- $P \in \mathcal{C}^1$ smooth implies $Q := \Phi(P) \in \mathcal{C}^2$ smooth
- $r_1 = r_2 = r$. After reordering $\Phi(\mathcal{C}^1_i) = \mathcal{C}^2_i$.
- $H_1(\mathbb{P}^2 \setminus \mathcal{C}^j_i; \mathbb{Z}) = \mathbb{Z}/\deg \mathcal{C}^j_i \Longrightarrow \deg \mathcal{C}^1_i = \deg \mathcal{C}^2_i$.

Local fundamental group

- $\pi_1^{\mathrm{loc}}(\mathbb{P}^2, \mathcal{C}^1, P) := \lim_{P \in V \text{ nghbd}} \pi_1(V \setminus \mathcal{C}^1)$
- $\pi_1^{\mathrm{loc}}(\mathbb{P}^2, \mathcal{C}^1, P)$ determines local topological type of $(\mathcal{C}^1, P)$ irreducible (Alexander polynomial, too).
- π_1^{loc} for $xy(x^p - y^q) = 0$, $\gcd(p, q) = 1$, is $\mathbb{F}_2 \times \mathbb{Z}$.



Topology and algebra of algebraic curves

Embedded homeomorphism types

- $\mathcal{C}^1, \mathcal{C}^2 \subset \mathbb{P}^2$, $\mathcal{C}^j = \mathcal{C}^j_1 \cup \cdots \cup \mathcal{C}^j_{r_j}$ irreducible decompositions.
- $\exists \Phi : (\mathbb{P}^2, \mathcal{C}^1) \to (\mathbb{P}^2, \mathcal{C}^2)$ homeomorphism?
- $P \in \mathcal{C}^1$ smooth implies $Q := \Phi(P) \in \mathcal{C}^2$ smooth
- $r_1 = r_2 = r$. After reordering $\Phi(\mathcal{C}^1_i) = \mathcal{C}^2_i$.
- $H_1(\mathbb{P}^2 \smallsetminus \mathcal{C}^j_i; \mathbb{Z}) = \mathbb{Z}/\deg \mathcal{C}^j_i \Longrightarrow \deg \mathcal{C}^1_i = \deg \mathcal{C}^2_i$.

Local fundamental group

- $\pi_1^{\mathrm{loc}}(\mathbb{P}^2, \mathcal{C}^1, P) := \lim_{P \in V \text{ nghbd}} \pi_1(V \smallsetminus \mathcal{C}^1)$
- $\pi_1^{\mathrm{loc}}(\mathbb{P}^2, \mathcal{C}^1, P)$ determines local topological type of $(\mathcal{C}^1, P)$ irreducible (Alexander polynomial, too).
- π_1^{loc} for $xy(x^p - y^q) = 0$, $\gcd(p, q) = 1$, is $\mathbb{F}_2 \times \mathbb{Z}$.
- π_1^{loc} with peripheral invariants does determine the local topological type of $(\mathcal{C}^1, P)$.



Topology and algebra of algebraic curves

Embedded homeomorphism types

- $\mathcal{C}^1, \mathcal{C}^2 \subset \mathbb{P}^2$, $\mathcal{C}^j = \mathcal{C}^j_1 \cup \cdots \cup \mathcal{C}^j_{r_j}$ irreducible decompositions.
- $\exists \Phi : (\mathbb{P}^2, \mathcal{C}^1) \to (\mathbb{P}^2, \mathcal{C}^2)$ homeomorphism?
- $P \in \mathcal{C}^1$ smooth implies $Q := \Phi(P) \in \mathcal{C}^2$ smooth
- $r_1 = r_2 = r$. After reordering $\Phi(\mathcal{C}^1_i) = \mathcal{C}^2_i$.
- $H_1(\mathbb{P}^2 \setminus \mathcal{C}^j_i; \mathbb{Z}) = \mathbb{Z}/\deg \mathcal{C}^j_i \Longrightarrow \deg \mathcal{C}^1_i = \deg \mathcal{C}^2_i$.
- $\mathcal{C}^1, \mathcal{C}^2$ have the same combinatorics.

Definition (Combinatorics)

$\mathcal{C} \subset \mathbb{P}^2$, $\sigma : X \to \mathbb{P}^2$ minimal embedded resolution of $\mathrm{Sing}^1(\mathcal{C})$ (take out nodes with branches in distinct global irreducible components).



Topology and algebra of algebraic curves

Embedded homeomorphism types

- $\mathcal{C}^1, \mathcal{C}^2 \subset \mathbb{P}^2$, $\mathcal{C}^j = \mathcal{C}^j_1 \cup \cdots \cup \mathcal{C}^j_{r_j}$ irreducible decompositions.
- $\exists \Phi : (\mathbb{P}^2, \mathcal{C}^1) \to (\mathbb{P}^2, \mathcal{C}^2)$ homeomorphism?
- $P \in \mathcal{C}^1$ smooth implies $Q := \Phi(P) \in \mathcal{C}^2$ smooth
- $r_1 = r_2 = r$. After reordering $\Phi(\mathcal{C}^1_i) = \mathcal{C}^2_i$.
- $H_1(\mathbb{P}^2 \setminus \mathcal{C}^j_i; \mathbb{Z}) = \mathbb{Z}/\deg \mathcal{C}^j_i \Longrightarrow \deg \mathcal{C}^1_i = \deg \mathcal{C}^2_i$.
- $\mathcal{C}^1, \mathcal{C}^2$ have the same combinatorics.

Definition (Combinatorics)

$\mathcal{C} \subset \mathbb{P}^2$, $\sigma : X \to \mathbb{P}^2$ minimal embedded resolution of $\mathrm{Sing}^1(\mathcal{C})$: $\sigma^{-1}(\mathcal{C})$ normal crossing divisor. Combinatorics:

- dual graph of $\sigma^{-1}(\mathcal{C})$;
- weighted with genus and self-intersections;
- strict transforms of irreducible components of $\mathcal{C}$ are marked.



Zariski tuples

Definition (Zariski pairs)

$\mathcal{C}^1, \mathcal{C}^2 \subset \mathbb{P}^2$ form a Zariski pair if they share combinatorics and $\nexists \Phi : (\mathbb{P}^2, \mathcal{C}^1) \to (\mathbb{P}^2, \mathcal{C}^2)$ homeomorphism.



Zariski tuples

Definition (Zariski pairs)

$\mathcal{C}^1, \mathcal{C}^2 \subset \mathbb{P}^2$ form a Zariski pair if they share combinatorics and $\nexists \Phi : (\mathbb{P}^2, \mathcal{C}^1) \to (\mathbb{P}^2, \mathcal{C}^2)$ homeomorphism.

Combinatorics and local topology

If $\mathcal{C}^1, \mathcal{C}^2 \subset \mathbb{P}^2$ share combinatorics $\exists \Phi : (\mathcal{T}(\mathcal{C}^1), \mathcal{C}^1) \to (\mathcal{T}(\mathcal{C}^2), \mathcal{C}^2)$ homeomorphism for suitable regular neighborhoods.



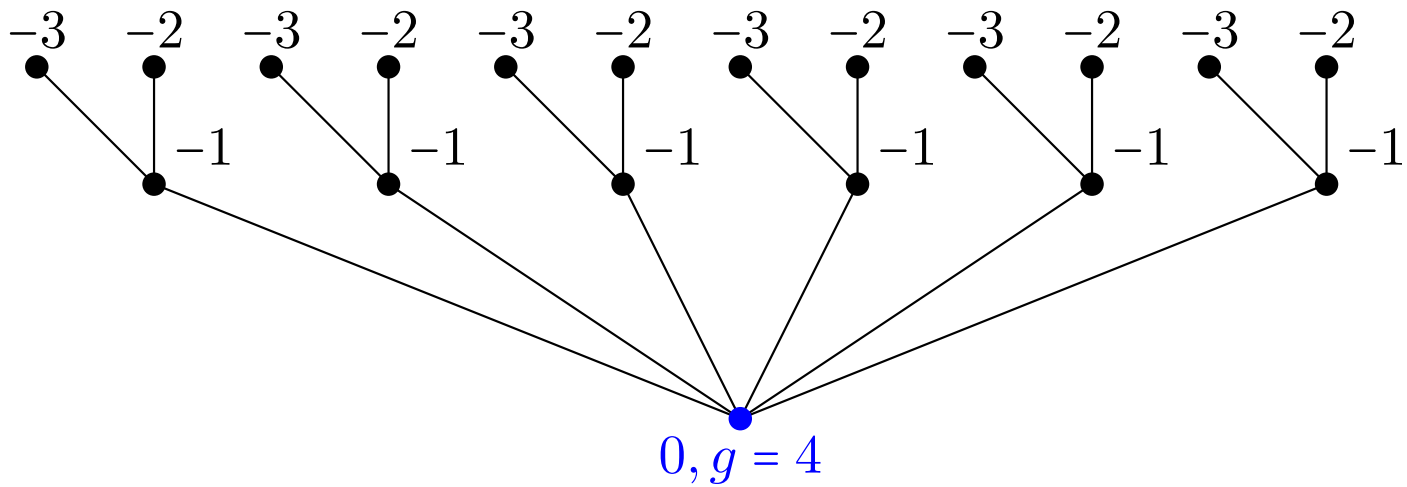
Zariski tuples

Definition (Zariski pairs)

$\mathcal{C}^1, \mathcal{C}^2 \subset \mathbb{P}^2$ form a Zariski pair if they share combinatorics and $\nexists \Phi : (\mathbb{P}^2, \mathcal{C}^1) \to (\mathbb{P}^2, \mathcal{C}^2)$ homeomorphism.

Combinatorics and local topology

If $\mathcal{C}^1, \mathcal{C}^2 \subset \mathbb{P}^2$ share combinatorics $\exists \Phi : (\mathcal{T}(\mathcal{C}^1), \mathcal{C}^1) \to (\mathcal{T}(\mathcal{C}^2), \mathcal{C}^2)$ homeomorphism for suitable regular neighborhoods.



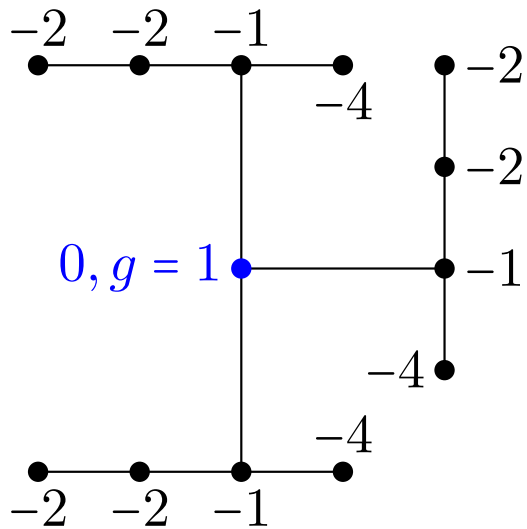
Zariski tuples

Definition (Zariski pairs)

$\mathcal{C}^1, \mathcal{C}^2 \subset \mathbb{P}^2$ form a Zariski pair if they share combinatorics and $\nexists \Phi : (\mathbb{P}^2, \mathcal{C}^1) \to (\mathbb{P}^2, \mathcal{C}^2)$ homeomorphism.

Combinatorics and local topology

If $\mathcal{C}^1, \mathcal{C}^2 \subset \mathbb{P}^2$ share combinatorics $\exists \Phi : (\mathcal{T}(\mathcal{C}^1), \mathcal{C}^1) \to (\mathcal{T}(\mathcal{C}^2), \mathcal{C}^2)$ homeomorphism for suitable regular neighborhoods.





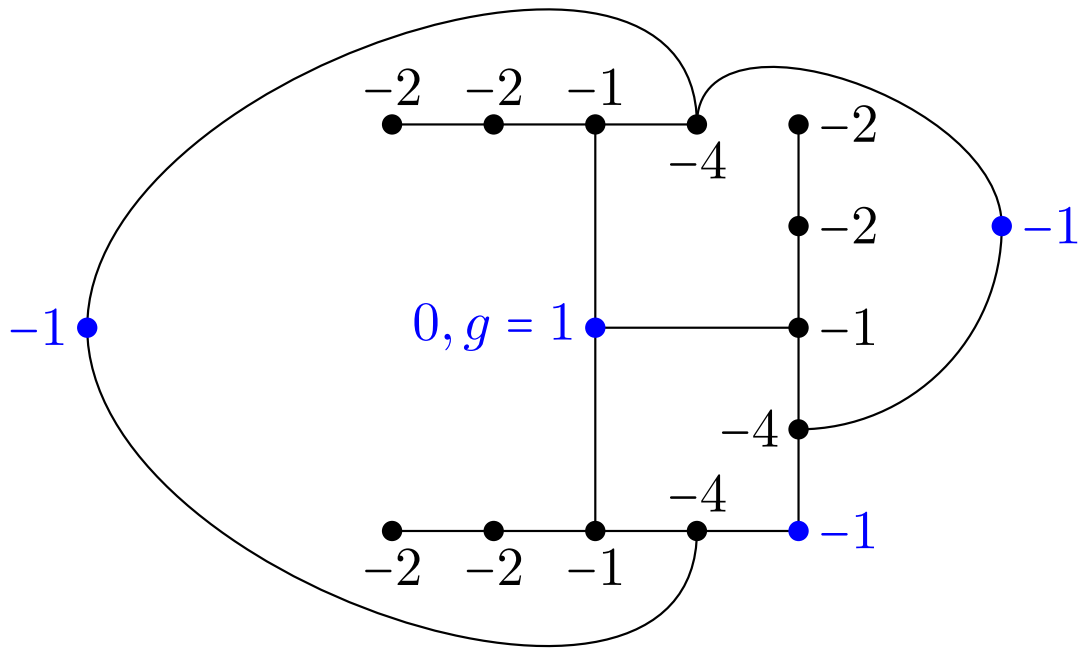
Zariski tuples

Definition (Zariski pairs)

$\mathcal{C}^1, \mathcal{C}^2 \subset \mathbb{P}^2$ form a Zariski pair if they share combinatorics and $\nexists \Phi : (\mathbb{P}^2, \mathcal{C}^1) \to (\mathbb{P}^2, \mathcal{C}^2)$ homeomorphism.

Combinatorics and local topology

If $\mathcal{C}^1, \mathcal{C}^2 \subset \mathbb{P}^2$ share combinatorics $\exists \Phi : (\mathcal{T}(\mathcal{C}^1), \mathcal{C}^1) \to (\mathcal{T}(\mathcal{C}^2), \mathcal{C}^2)$ homeomorphism for suitable regular neighborhoods.



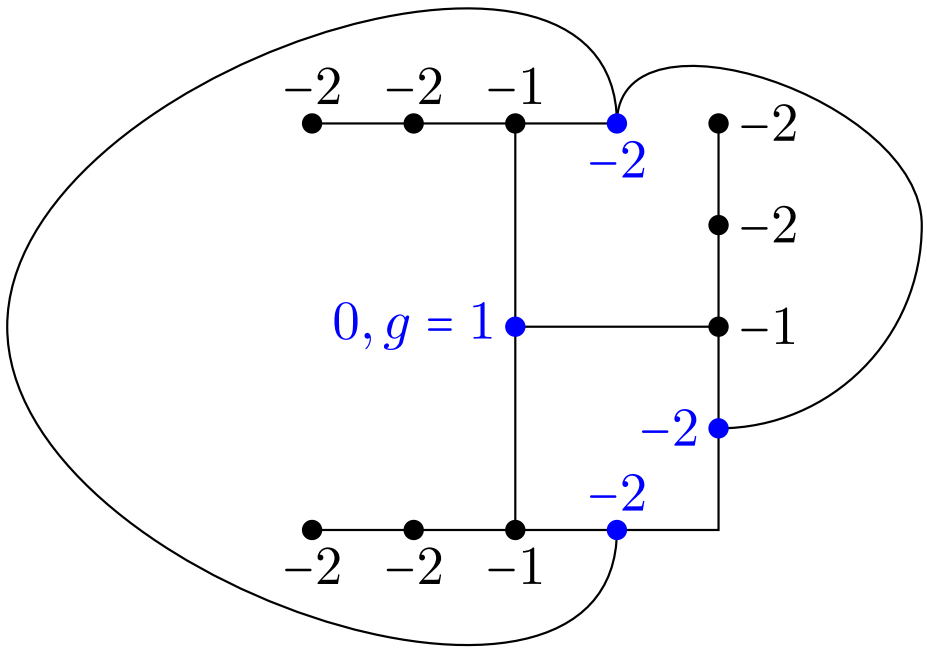
Zariski tuples

Definition (Zariski pairs)

$\mathcal{C}^1, \mathcal{C}^2 \subset \mathbb{P}^2$ form a Zariski pair if they share combinatorics and $\nexists \Phi : (\mathbb{P}^2, \mathcal{C}^1) \to (\mathbb{P}^2, \mathcal{C}^2)$ homeomorphism.

Combinatorics and local topology

If $\mathcal{C}^1, \mathcal{C}^2 \subset \mathbb{P}^2$ share combinatorics $\exists \Phi : (\mathcal{T}(\mathcal{C}^1), \mathcal{C}^1) \to (\mathcal{T}(\mathcal{C}^2), \mathcal{C}^2)$ homeomorphism for suitable regular neighborhoods.



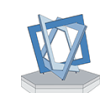
Techniques and examples

- Sextics with six cusps on a conic (or not) (Zariski, Oka, _)



Techniques and examples

- Sextics with six cusps on a conic (or not) (Zariski, Oka, _)
  - $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_i)$: $\mathbb{Z}/2 * \mathbb{Z}/3$ (on a conic), $\mathbb{Z}/2 \times \mathbb{Z}/3$ (not on a conic).



Techniques and examples

- Sextics with six cusps on a conic (or not) (Zariski, Oka, __)
  - $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_i)$: $\mathbb{Z}/2 * \mathbb{Z}/3$ (on a conic), $\mathbb{Z}/2 \times \mathbb{Z}/3$ (not on a conic).
  - $\Delta = t^2 - t + 1$ or $\Delta = 1$ (not on a conic), algebraic computations.



Techniques and examples

- Sextics with 3 $\mathbb{E}_6$ on a conic (or not) (__, Carmona)
  - $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_i)$: $\mathbb{Z}/2 * \mathbb{Z}/3$ (on a conic), $\mathbb{Z}/2 \times \mathbb{Z}/3$ (not on a conic).
  - $\Delta = t^2 - t + 1$ or $\Delta = 1$ (not on a conic), algebraic computations.



Techniques and examples

- Sextics with 3 $\mathbb{E}_6$ on a conic (or not) (_, Carmona)
- Alexander-equivalent cubics and conics using fundamental group (_, Carmona).



Techniques and examples

- Sextics with 3 $\mathbb{E}_6$ on a conic (or not) (_, Carmona)
- Alexander-equivalent cubics and conics using fundamental group (_, Carmona).
  - Distinguished by characteristic varieties (_, Cogolludo, Matei)



Techniques and examples

- Sextics with 3 $\mathbb{E}_6$ on a conic (or not) (_, Carmona)
- Alexander-equivalent cubics and conics using fundamental group (_, Carmona).
  - Distinguished by characteristic varieties (_, Cogolludo, Matei)
- Sextic case classified by Degtyarev and Shimada: using $K3$ lattice arithmetic.



Techniques and examples

- Sextics with 3 $\mathbb{E}_6$ on a conic (or not) (_, Carmona)
- Alexander-equivalent cubics and conics using fundamental group (_, Carmona).
  - Distinguished by characteristic varieties (_, Cogolludo, Matei)
- Sextic case classified by Degtyarev and Shimada: using $K3$ lattice arithmetic.
- Cyclic covers



Techniques and examples

- Sextics with 3 $\mathbb{E}_6$ on a conic (or not) (_, Carmona)
- Alexander-equivalent cubics and conics using fundamental group (_, Carmona).
  - Distinguished by characteristic varieties (_, Cogolludo, Matei)
- Sextic case classified by Degtyarev and Shimada: using $K3$ lattice arithmetic.
- Dihedral covers by algebraic arguments (Tokunaga)



Techniques and examples

- Sextics with 3 $\mathbb{E}_6$ on a conic (or not) (_, Carmona)
- Alexander-equivalent cubics and conics using fundamental group (_, Carmona).
  - Distinguished by characteristic varieties (_, Cogolludo, Matei)
- Sextic case classified by Degtyarev and Shimada: using $K3$ lattice arithmetic.
- Dihedral covers by algebraic arguments (Tokunaga)
- Line arrangements: Rybnikov (13 lines), $\neq$ fundamental group.
  - Truncated Alexander invariants (_, Carmona, Cogolludo, Marco).



Techniques and examples

- Sextics with 3 $\mathbb{E}_6$ on a conic (or not) (_, Carmona)
- Alexander-equivalent cubics and conics using fundamental group (_, Carmona).
  - Distinguished by characteristic varieties (_, Cogolludo, Matei)
- Sextic case classified by Degtyarev and Shimada: using $K3$ lattice arithmetic.
- Dihedral covers by algebraic arguments (Tokunaga)
- Line arrangements: Rybnikov (13 lines), $\neq$ fundamental group.
  - Truncated Alexander invariants (_, Carmona, Cogolludo, Marco).
  - Based on homological rigidity (Marco).



Techniques and examples

- Sextics with 3 $\mathbb{E}_6$ on a conic (or not) (_, Carmona)
- Alexander-equivalent cubics and conics using fundamental group (_, Carmona).
  - Distinguished by characteristic varieties (_, Cogolludo, Matei)
- Sextic case classified by Degtyarev and Shimada: using $K3$ lattice arithmetic.
- Dihedral covers by algebraic arguments (Tokunaga)
- Line arrangements: Rybnikov (13 lines), $\neq$ fundamental group.
  - Truncated Alexander invariants (_, Carmona, Cogolludo, Marco).
  - Based on homological rigidity (Marco).
- Braid monodromy (_, Carmona, Cogolludo)



Techniques and examples

- Sextics with 3 $\mathbb{E}_6$ on a conic (or not) (_, Carmona)
- Alexander-equivalent cubics and conics using fundamental group (_, Carmona).
  - Distinguished by characteristic varieties (_, Cogolludo, Matei)
- Sextic case classified by Degtyarev and Shimada: using $K3$ lattice arithmetic.
- Dihedral covers by algebraic arguments (Tokunaga)
- Line arrangements: Rybnikov (13 lines), $\neq$ fundamental group.
  - Truncated Alexander invariants (_, Carmona, Cogolludo, Marco).
  - Based on homological rigidity (Marco).
- Braid monodromy (_, Carmona, Cogolludo)

Definition

A Zariski tuple is an arithmetic Zariski tuple if the equations of the curves are Galois-conjugate.



Techniques and examples

- Sextics with 3 $\mathbb{E}_6$ on a conic (or not) (_, Carmona)
- Alexander-equivalent cubics and conics using fundamental group (_, Carmona).
  - Distinguished by characteristic varieties (_, Cogolludo, Matei)
- Sextic case classified by Degtyarev and Shimada: using $K3$ lattice arithmetic.
- Dihedral covers by algebraic arguments (Tokunaga)
- Line arrangements: Rybnikov (13 lines), $\neq$ fundamental group.
  - Truncated Alexander invariants (_, Carmona, Cogolludo, Marco).
  - Based on homological rigidity (Marco).
- Braid monodromy (_, Carmona, Cogolludo)

Definition

A Zariski tuple is an arithmetic Zariski tuple if the equations of the curves are Galois-conjugate.

Purely algebraic methods cannot work!



AZT and groups

- AZP for arrangements of lines in $\mathbb{Q}(\sqrt{5})$ with braid monodromy (_, Carmona, Cogolludo, Marco): no information about π_1.



AZT and groups

- AZP for arrangements of lines in $\mathbb{Q}(\sqrt{5})$ with braid monodromy (_, Carmona, Cogolludo, Marco): no information about π_1.
- AZP with homeomorphic complements (_, Cogolludo).



AZT and groups

- AZP for arrangements of lines in $\mathbb{Q}(\sqrt{5})$ with braid monodromy (_, Carmona, Cogolludo, Marco): no information about π_1.
- AZP with homeomorphic complements (_, Cogolludo).
- AZP of line arrangements with isomorphic π_1 (Guerville).



AZT and groups

- AZP for arrangements of lines in $\mathbb{Q}(\sqrt{5})$ with braid monodromy (__, Carmona, Cogolludo, Marco): no information about π_1.
- AZP with homeomorphic complements (__, Cogolludo).
- AZP of line arrangements with isomorphic π_1 (Guerville).
- AZP in $\mathbb{Q}(\zeta_5)$ (of line arrangements) with non-isomorphic π_1 (__, Cogolludo, Guerville, Marco) via truncated Alexander invariants.



AZT and groups

- AZP for arrangements of lines in $\mathbb{Q}(\sqrt{5})$ with braid monodromy (__, Carmona, Cogolludo, Marco): no information about π_1.
- AZP with homeomorphic complements (__, Cogolludo).
- AZP of line arrangements with isomorphic π_1 (Guerville).
- AZP in $\mathbb{Q}(\zeta_5)$ (of line arrangements) with non-isomorphic π_1 (__, Cogolludo, Guerville, Marco) via truncated Alexander invariants.
  - Above example first found by Guerville: distinguished via *linking invariant*.



AZT and groups

- AZP for arrangements of lines in $\mathbb{Q}(\sqrt{5})$ with braid monodromy (_, Carmona, Cogolludo, Marco): no information about π_1.
- AZP with homeomorphic complements (_, Cogolludo).
- AZP of line arrangements with isomorphic π_1 (Guerville).
- AZP in $\mathbb{Q}(\zeta_5)$ (of line arrangements) with non-isomorphic π_1 (_, Cogolludo, Guerville, Marco) via truncated Alexander invariants.
  - Above example first found by Guerville: distinguished via *linking invariant*.
  - Linking invariant developed by _, Florens, Guerville for line arrangements



AZT and groups

- AZP for arrangements of lines in $\mathbb{Q}(\sqrt{5})$ with braid monodromy (_, Carmona, Cogolludo, Marco): no information about π_1.
- AZP with homeomorphic complements (_, Cogolludo).
- AZP of line arrangements with isomorphic π_1 (Guerville).
- AZP in $\mathbb{Q}(\zeta_5)$ (of line arrangements) with non-isomorphic π_1 (_, Cogolludo, Guerville, Marco) via truncated Alexander invariants.
  - Above example first found by Guerville: distinguished via *linking invariant*.
  - Linking invariant developed by _, Florens, Guerville for line arrangements
  - Generalized to arbitrary curves by Guerville, Meilhan, Shirane.



AZT and groups

- AZP for arrangements of lines in $\mathbb{Q}(\sqrt{5})$ with braid monodromy (_, Carmona, Cogolludo, Marco): no information about π_1.
- AZP with homeomorphic complements (_, Cogolludo).
- AZP of line arrangements with isomorphic π_1 (Guerville).
- AZP in $\mathbb{Q}(\zeta_5)$ (of line arrangements) with non-isomorphic π_1 (_, Cogolludo, Guerville, Marco) via truncated Alexander invariants.
  - Above example first found by Guerville: distinguished via *linking invariant*.
  - Linking invariant developed by _, Florens, Guerville for line arrangements
  - Generalized to arbitrary curves by Guerville, Meilhan, Shirane.
- Zariski pairs of complexified real arrangements by Guerville and Viu (algebraic properties, conics passing throught multiple points or tangent to lines).



AZT and groups

- AZP for arrangements of lines in $\mathbb{Q}(\sqrt{5})$ with braid monodromy (_, Carmona, Cogolludo, Marco): no information about π_1.
- AZP with homeomorphic complements (_, Cogolludo).
- AZP of line arrangements with isomorphic π_1 (Guerville).
- AZP in $\mathbb{Q}(\zeta_5)$ (of line arrangements) with non-isomorphic π_1 (_, Cogolludo, Guerville, Marco) via truncated Alexander invariants.
  - Above example first found by Guerville: distinguished via *linking invariant*.
  - Linking invariant developed by _, Florens, Guerville for line arrangements
  - Generalized to arbitrary curves by Guerville, Meilhan, Shirane.
- Zariski pairs of complexified real arrangements by Guerville and Viu (algebraic properties, conics passing throught multiple points or tangent to lines).
  - Fundamental groups are not equivalent via LCS (_, Guerville, Viu).



AZT and groups

- AZP for arrangements of lines in $\mathbb{Q}(\sqrt{5})$ with braid monodromy (_, Carmona, Cogolludo, Marco): no information about π_1.
- AZP with homeomorphic complements (_, Cogolludo).
- AZP of line arrangements with isomorphic π_1 (Guerville).
- AZP in $\mathbb{Q}(\zeta_5)$ (of line arrangements) with non-isomorphic π_1 (_, Cogolludo, Guerville, Marco) via truncated Alexander invariants.
  - Above example first found by Guerville: distinguished via *linking invariant*.
  - Linking invariant developed by _, Florens, Guerville for line arrangements
  - Generalized to arbitrary curves by Guerville, Meilhan, Shirane.
- Zariski pairs of complexified real arrangements by Guerville and Viu (algebraic properties, conics passing throught multiple points or tangent to lines).
  - Fundamental groups are not equivalent via LCS (_, Guerville, Viu).
- Linking invariant used by many authors.



Complements and fundamental groups

Oka-Eyral curves

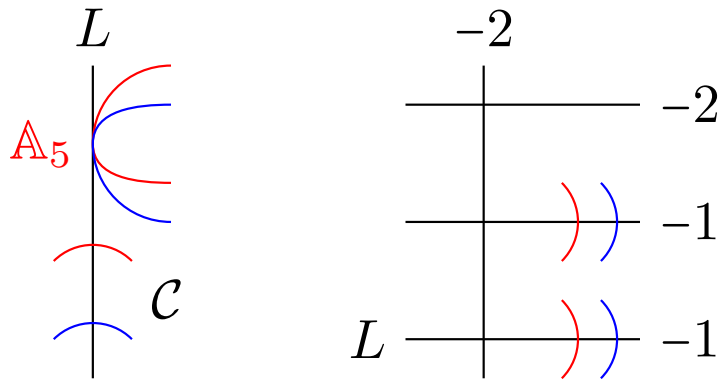
Arithmetic Zariski pair with isomorphic complements ($\mathbb{Q}(\sqrt{3})$)

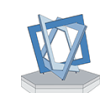


Complements and fundamental groups

Oka-Eyral curves

Arithmetic Zariski pair with isomorphic complements ($\mathbb{Q}(\sqrt{3})$)





Complements and fundamental groups

Oka-Eyral curves

Arithmetic Zariski pair with isomorphic complements ($\mathbb{Q}(\sqrt{3})$)

Theorem (_,Cogolludo, Martín)

In most cases if $\mathcal{C}_1, \mathcal{C}_2$ form a Zariski pair then $\mathbb{P}^2 \setminus \mathcal{C}_i$ are not homeomorphic.

Key of the proof.

Waldhausen: classification of graph manifolds and isomorphisms of fundamental groups of 3-manifold groups (sufficiently large). □



Complements and fundamental groups

Oka-Eyral curves

Arithmetic Zariski pair with isomorphic complements ($\mathbb{Q}(\sqrt{3})$)

Theorem (_,Cogolludo, Martín)

In most cases if $\mathcal{C}_1, \mathcal{C}_2$ form a Zariski pair then $\mathbb{P}^2 \setminus \mathcal{C}_i$ are not homeomorphic.

Key of the proof.

Waldhausen: classification of graph manifolds and isomorphisms of fundamental groups of 3-manifold groups (sufficiently large). □

Theorem (_,Cogolludo, Martín)

There are Zariski tuples (distinguished by linking invariant) with abelian fundamental groups.



Realization space I

$\mathscr{M}^{2d}_{3(d,d+1)}$

Set of projective plane curves in $\mathbb{P}^2 := \mathbb{CP}^2$ of degree $2d$ with three singular types of topological type as $v^d - u^{d+1} = 0$.



Realization space I

$\mathscr{M}^{2d}_{3(d,d+1)}$

Set of projective plane curves in $\mathbb{P}^2 := \mathbb{CP}^2$ of degree $2d$ with three singular types of topological type as $v^d - u^{d+1} = 0$.

$\mathcal{C} \in \mathscr{M}^{2d}_{3(d,d+1)}$



Realization space I

$\mathscr{M}^{2d}_{3(d,d+1)}$

Set of projective plane curves in $\mathbb{P}^2 := \mathbb{CP}^2$ of degree $2d$ with three singular types of topological type as $v^d - u^{d+1} = 0$.

$\mathcal{C} \in \mathscr{M}^{2d}_{3(d,d+1)}$

- $\mathrm{Sing}(\mathcal{C}) = \{P, Q, R\}$, $\mathcal{L}$ line $P, Q \in \mathcal{L}$.



Realization space I

$\mathscr{M}^{2d}_{3(d,d+1)}$

Set of projective plane curves in $\mathbb{P}^2 := \mathbb{CP}^2$ of degree $2d$ with three singular types of topological type as $v^d - u^{d+1} = 0$.

$\mathcal{C} \in \mathscr{M}^{2d}_{3(d,d+1)}$

- $\mathrm{Sing}(\mathcal{C}) = \{P, Q, R\}$, $\mathcal{L}$ line $P, Q \in \mathcal{L}$.
- $\mathcal{C}$ irreducible: all its singular points locally irreducible.



Realization space I

$\mathscr{M}^{2d}_{3(d,d+1)}$

Set of projective plane curves in $\mathbb{P}^2 := \mathbb{CP}^2$ of degree $2d$ with three singular types of topological type as $v^d - u^{d+1} = 0$.

$\mathcal{C} \in \mathscr{M}^{2d}_{3(d,d+1)}$

- $\mathrm{Sing}(\mathcal{C}) = \{P, Q, R\}$, $\mathcal{L}$ line $P, Q \in \mathcal{L}$.
- $\mathcal{C}$ irreducible: all its singular points locally irreducible.
- $\mathcal{C} \cdot \mathcal{L} = 2d$, $(\mathcal{C} \cdot \mathcal{L})_P, (\mathcal{C} \cdot \mathcal{L})_Q \geq d \Longrightarrow \begin{cases} (\mathcal{C} \cdot \mathcal{L})_P = (\mathcal{C} \cdot \mathcal{L})_Q = d \\ \mathcal{C} \cap \mathcal{L} = \{P, Q\}. \end{cases}$



Realization space I

$\mathscr{M}^{2d}_{3(d,d+1)}$

Set of projective plane curves in $\mathbb{P}^2 := \mathbb{CP}^2$ of degree $2d$ with three singular types of topological type as $v^d - u^{d+1} = 0$.

$\mathcal{C} \in \mathscr{M}^{2d}_{3(d,d+1)}$

- $\mathrm{Sing}(\mathcal{C}) = \{P, Q, R\}$, $\mathcal{L}$ line $P, Q \in \mathcal{L}$.
- $\mathcal{C}$ irreducible: all its singular points locally irreducible.
- $\mathcal{C} \cdot \mathcal{L} = 2d$, $(\mathcal{C} \cdot \mathcal{L})_P, (\mathcal{C} \cdot \mathcal{L})_Q \geq d \Longrightarrow \begin{cases} (\mathcal{C} \cdot \mathcal{L})_P = (\mathcal{C} \cdot \mathcal{L})_Q = d \\ \mathcal{C} \cap \mathcal{L} = \{P, Q\}. \end{cases}$

Special form



Realization space I

$\mathscr{M}^{2d}_{3(d,d+1)}$

Set of projective plane curves in $\mathbb{P}^2 := \mathbb{CP}^2$ of degree $2d$ with three singular types of topological type as $v^d - u^{d+1} = 0$.

$\mathcal{C} \in \mathscr{M}^{2d}_{3(d,d+1)}$

- $\mathrm{Sing}(\mathcal{C}) = \{P, Q, R\}$, $\mathcal{L}$ line $P, Q \in \mathcal{L}$.
- $\mathcal{C}$ irreducible: all its singular points locally irreducible.
- $\mathcal{C} \cdot \mathcal{L} = 2d$, $(\mathcal{C} \cdot \mathcal{L})_P, (\mathcal{C} \cdot \mathcal{L})_Q \geq d \Longrightarrow \begin{cases} (\mathcal{C} \cdot \mathcal{L})_P = (\mathcal{C} \cdot \mathcal{L})_Q = d \\ \mathcal{C} \cap \mathcal{L} = \{P, Q\}. \end{cases}$

Special form

- Up to $\mathrm{PGL}(3;\mathbb{C})$, $\mathrm{Sing}(\mathcal{C}) = \{[1:0:0], [0:1:0], [0:0:1]\}$.



Realization space I

$\mathscr{M}^{2d}_{3(d,d+1)}$

Set of projective plane curves in $\mathbb{P}^2 := \mathbb{CP}^2$ of degree $2d$ with three singular types of topological type as $v^d - u^{d+1} = 0$.

$\mathcal{C} \in \mathscr{M}^{2d}_{3(d,d+1)}$

- $\mathrm{Sing}(\mathcal{C}) = \{P, Q, R\}$, $\mathcal{L}$ line $P, Q \in \mathcal{L}$.
- $\mathcal{C}$ irreducible: all its singular points locally irreducible.
- $\mathcal{C} \cdot \mathcal{L} = 2d$, $(\mathcal{C} \cdot \mathcal{L})_P, (\mathcal{C} \cdot \mathcal{L})_Q \geq d \Longrightarrow \begin{cases} (\mathcal{C} \cdot \mathcal{L})_P = (\mathcal{C} \cdot \mathcal{L})_Q = d \\ \mathcal{C} \cap \mathcal{L} = \{P, Q\}. \end{cases}$

Special form

- Up to $\mathrm{PGL}(3;\mathbb{C})$, $\mathrm{Sing}(\mathcal{C}) = \{[1:0:0], [0:1:0], [0:0:1]\}$.
- Use diagonal-permutation automorphisms.



Realization Space II

Finding equations

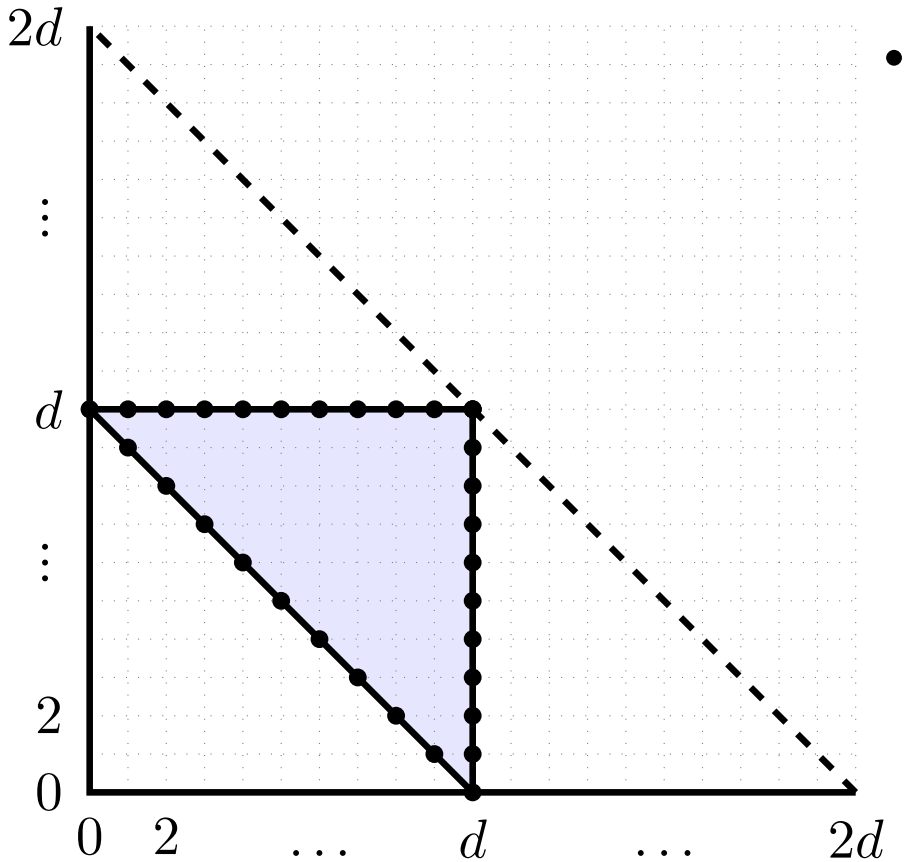
$\mathcal{C} = \{F_{2d}(X, Y, Z) = 0\}$, $Z = 0$ not tangent line for $(\mathcal{C}, [0 : 1 : 0]) \Longrightarrow$ coefficient of $(XY)^d$ is $\neq 0$ ($\Rightarrow$ is 1)



Realization Space II

Finding equations

$\mathcal{C} = \{F_{2d}(X, Y, Z) = 0\}$, $Z = 0$ not tangent line for $(\mathcal{C}, [0 : 1 : 0]) \Longrightarrow$ coefficient of $(XY)^d$ is $\neq 0$ ($\Rightarrow$ is 1)

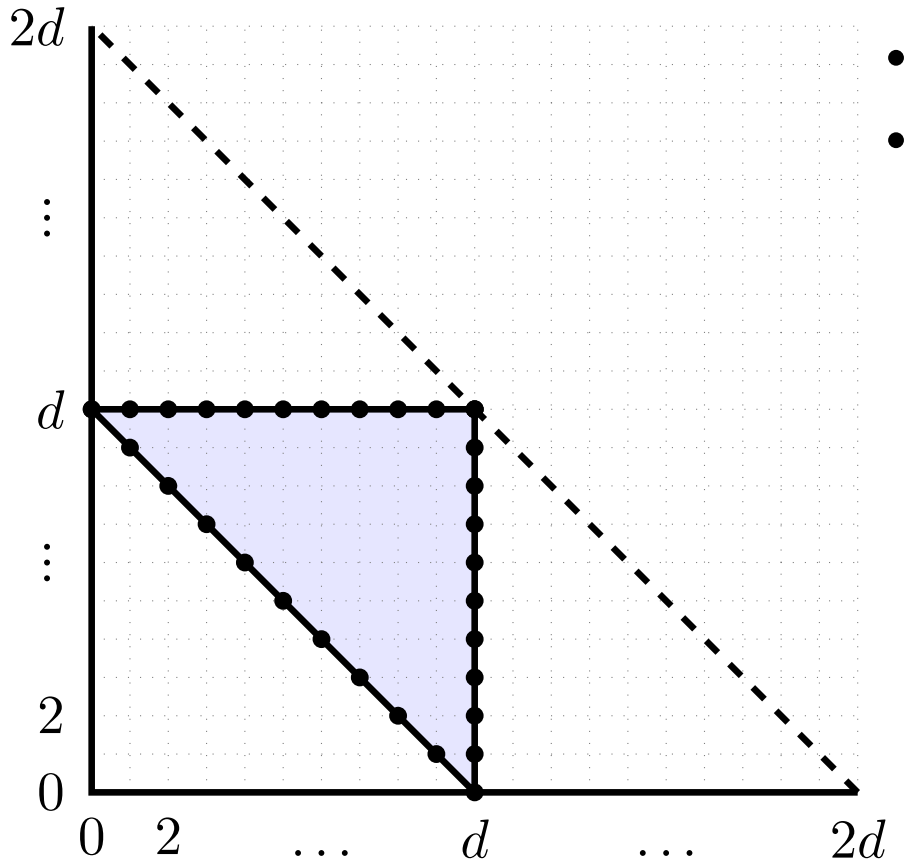


- Newton polygon of $F(X, Y, 1)$.

Realization Space II

Finding equations

$\mathcal{C} = \{F_{2d}(X, Y, Z) = 0\}$, $Z = 0$ not tangent line for $(\mathcal{C}, [0 : 1 : 0]) \Longrightarrow$ coefficient of $(XY)^d$ is $\neq 0$ ($\Rightarrow$ is 1)



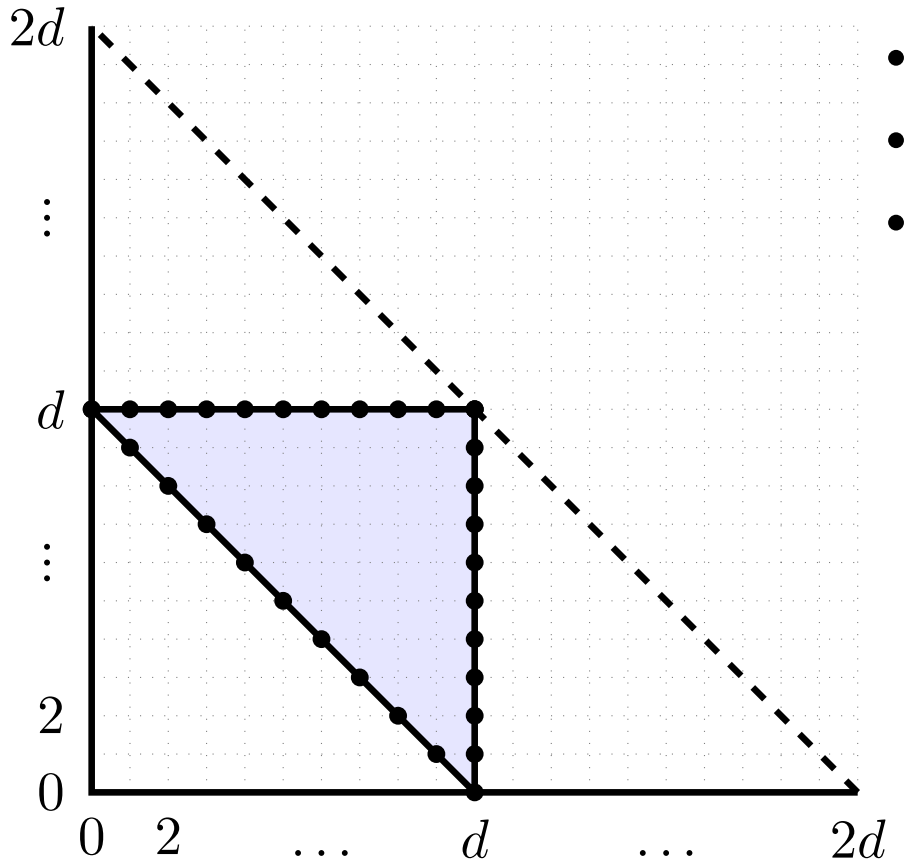
- Newton polygon of $F(X, Y, 1)$.
- One tangent line $\neq X, Y$



Realization Space II

Finding equations

$\mathcal{C} = \{F_{2d}(X, Y, Z) = 0\}$, $Z = 0$ not tangent line for $(\mathcal{C}, [0 : 1 : 0]) \Longrightarrow$ coefficient of $(XY)^d$ is $\neq 0$ ($\Rightarrow$ is 1)



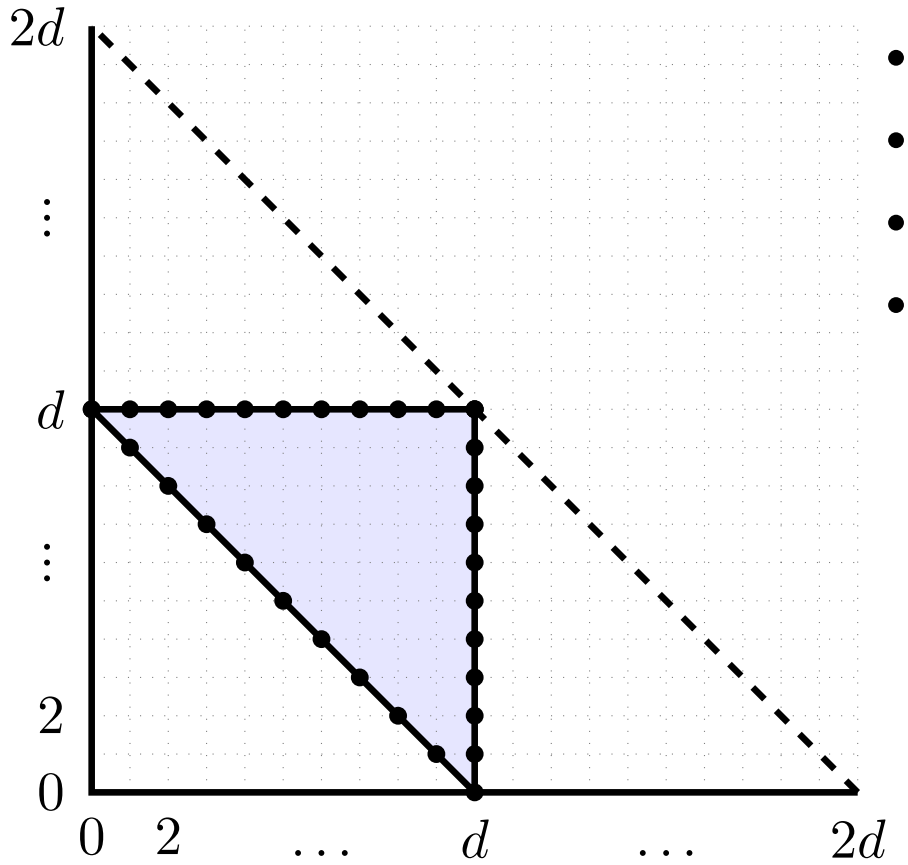
- Newton polygon of $F(X, Y, 1)$.
- One tangent line $\neq X, Y$
- Apply to $F(X, 1, Z), F(1, Y, Z)$



Realization Space II

Finding equations

$\mathcal{C} = \{F_{2d}(X, Y, Z) = 0\}$, $Z = 0$ not tangent line for $(\mathcal{C}, [0 : 1 : 0]) \Longrightarrow$ coefficient of $(XY)^d$ is $\neq 0$ ($\Rightarrow$ is 1)



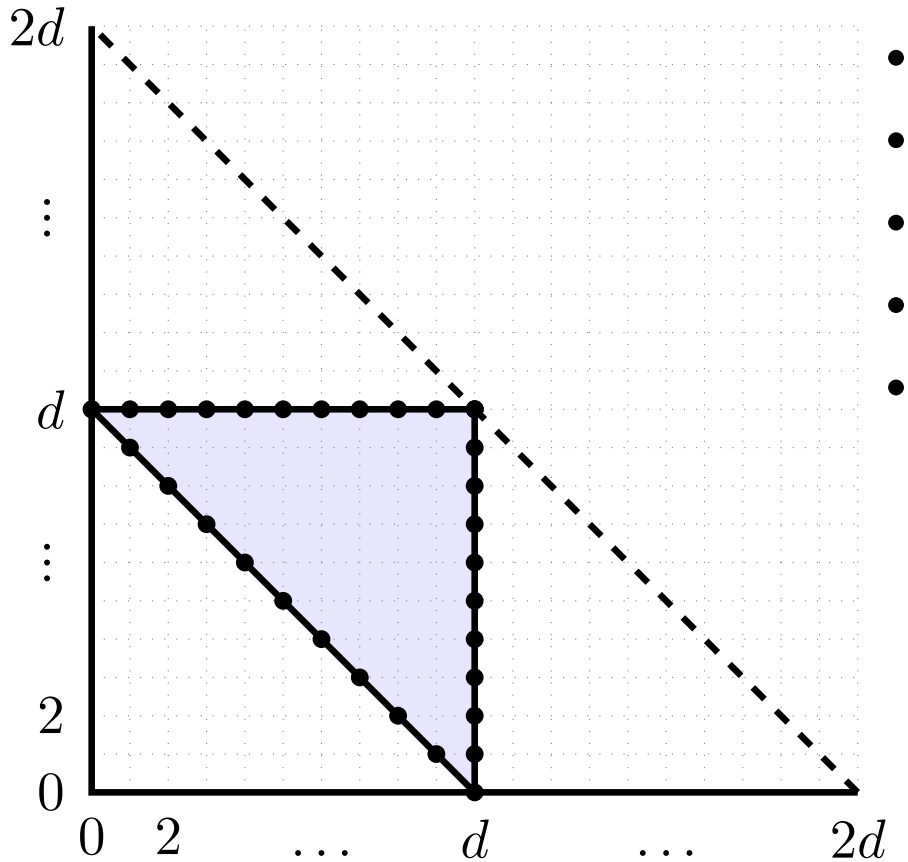
- Newton polygon of $F(X, Y, 1)$.
- One tangent line $\neq X, Y$
- Apply to $F(X, 1, Z), F(1, Y, Z)$
- $Y^d(X + \alpha Z)^d \mapsto Y^d(X + Z)^d$



Realization Space II

Finding equations

$\mathcal{C} = \{F_{2d}(X, Y, Z) = 0\}$, $Z = 0$ not tangent line for $(\mathcal{C}, [0 : 1 : 0]) \Longrightarrow$ coefficient of $(XY)^d$ is $\neq 0$ ($\Rightarrow$ is 1)



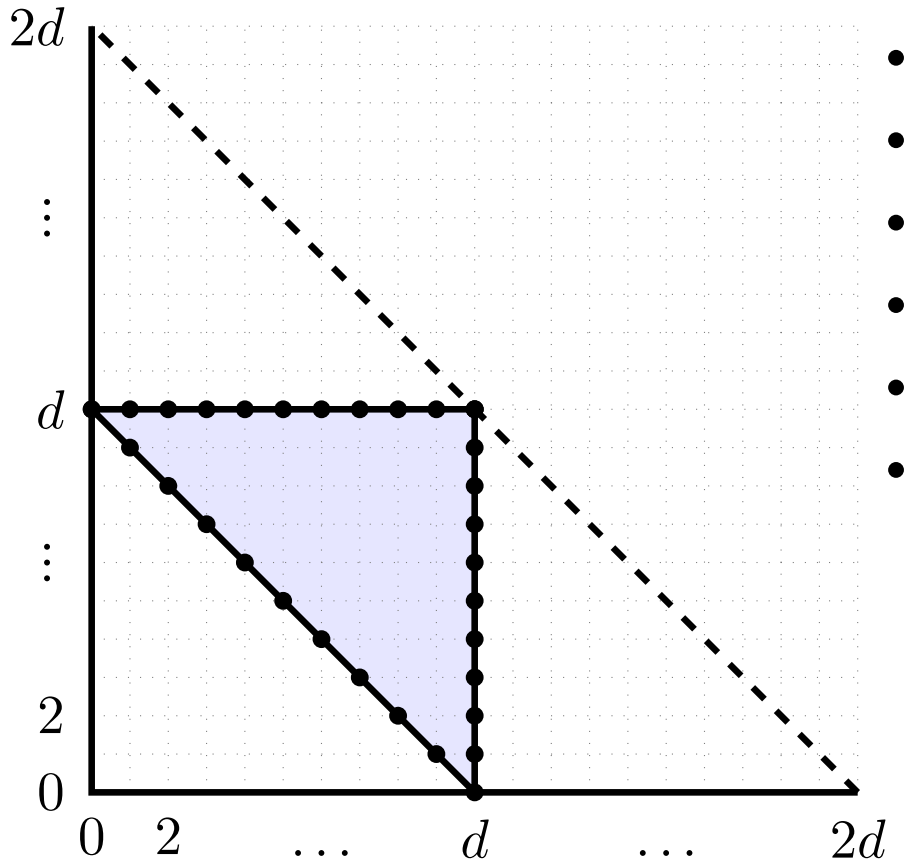
- Newton polygon of $F(X, Y, 1)$.
- One tangent line $\neq X, Y$
- Apply to $F(X, 1, Z), F(1, Y, Z)$
- $Y^d(X + \alpha Z)^d \mapsto Y^d(X + Z)^d$
- $X^d(Y + \alpha Z)^d \mapsto X^d(Y + Z)^d$



Realization Space II

Finding equations

$\mathcal{C} = \{F_{2d}(X, Y, Z) = 0\}$, $Z = 0$ not tangent line for $(\mathcal{C}, [0 : 1 : 0]) \Longrightarrow$ coefficient of $(XY)^d$ is $\neq 0$ ($\Rightarrow$ is 1)



- Newton polygon of $F(X, Y, 1)$.
- One tangent line $\neq X, Y$
- Apply to $F(X, 1, Z), F(1, Y, Z)$
- $Y^d(X + \alpha Z)^d \mapsto Y^d(X + Z)^d$
- $X^d(Y + \alpha Z)^d \mapsto X^d(Y + Z)^d$
- $Z^d(X + \omega Y)^d, \omega^d = 1$.



Realization Space III

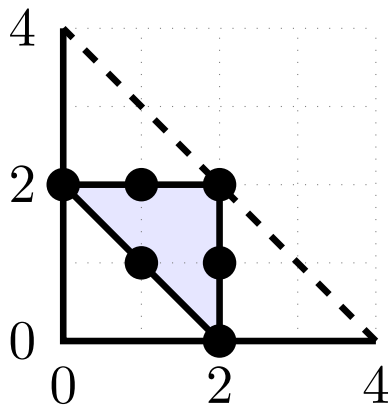
Decomposition via roots of unity

$\mathscr{M}^{2d}_{3(d,d+1)}$ decomposes in $\left\lfloor \frac{d}{2} \right\rfloor + 1$ subsets $\mathscr{M}^{2d}_{3(d,d+1)}(\omega)$ parametrized by the sets $\{\omega, \omega^{-1}\}$, when $\omega^d = 1$.



Small degrees

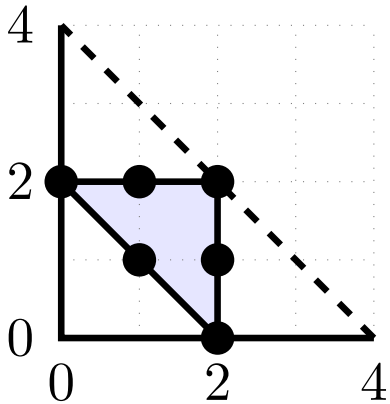
$d = 2$





Small degrees

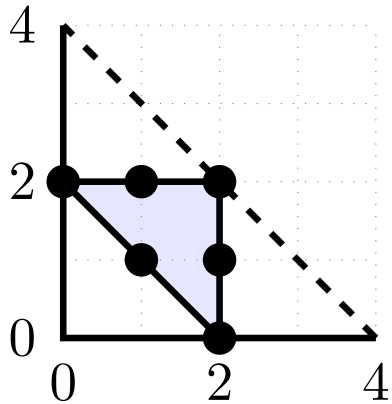
$d = 2$



$\boxed{\omega = 1}$ $(YZ + XZ + XY)^2$, $\nexists \mathcal{C}$

Small degrees

$d = 2$



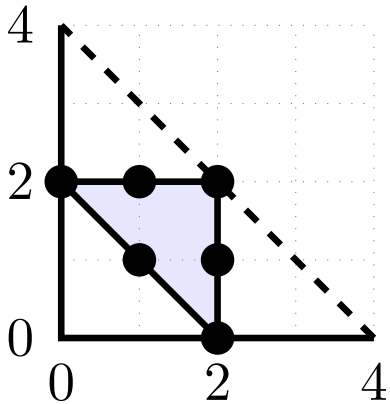
$\boxed{\omega = 1}$ $(YZ + XZ + XY)^2$, $\nexists \mathcal{C}$

$\boxed{\omega = -1}$ $\mathcal{C}$ tricuspidal quartic



Small degrees

$d = 2$



$\boxed{\omega = 1}$ $(YZ + XZ + XY)^2$, $\nexists \mathcal{C}$

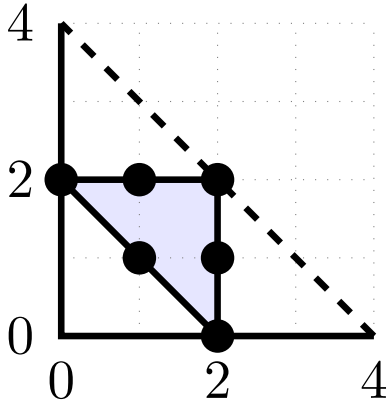
$\boxed{\omega = -1}$ $\mathcal{C}$ tricuspidal quartic

Zariski: $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ non-abelian



Small degrees

$d = 2$

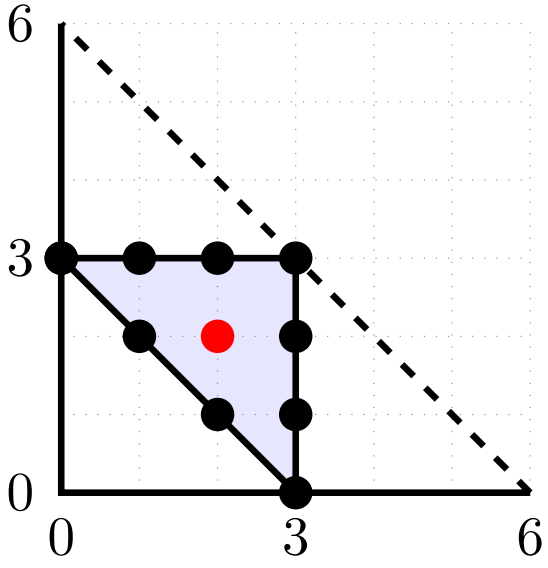


$\boxed{\omega = 1}$ $(YZ + XZ + XY)^2$, $\nexists \mathcal{C}$

$\boxed{\omega = -1}$ $\mathcal{C}$ tricuspidal quartic

Zariski: $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ non-abelian

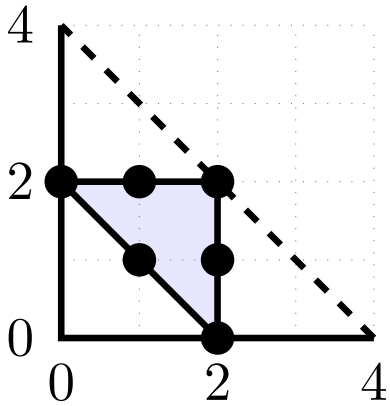
$d = 3$





Small degrees

$d = 2$

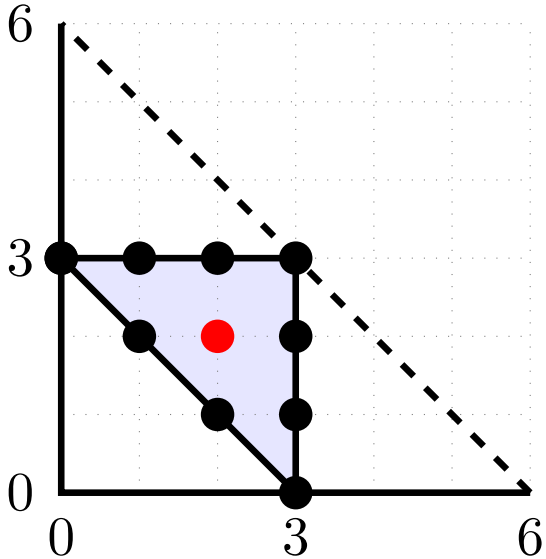


$\boxed{\omega = 1}$ $(YZ + XZ + XY)^2$, $\nexists \mathcal{C}$

$\boxed{\omega = -1}$ $\mathcal{C}$ tricuspidal quartic

Zariski: $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ non-abelian

$d = 3$

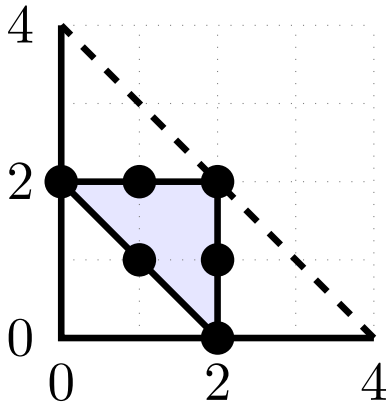


$\boxed{\omega = 1}$ $\exists$ conic tangent to $\mathcal{C}_1$



Small degrees

$d = 2$

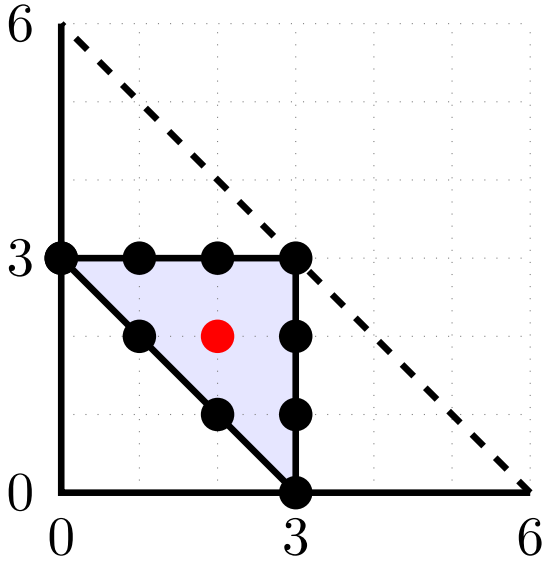


$\boxed{\omega = 1}$ $(YZ + XZ + XY)^2$, $\nexists \mathcal{C}$

$\boxed{\omega = -1}$ $\mathcal{C}$ tricuspidal quartic

Zariski: $\pi_1(\mathbb{P}^2 \setminus \mathcal{C})$ non-abelian

$d = 3$



$\boxed{\omega = 1}$ $\exists$ conic tangent to $\mathcal{C}_1$

$\boxed{\omega = \zeta_3}$ $\mathcal{C}_{\zeta_3}$, $\nexists$ such a conic



Realization space

Theorem

If $d \geq 3$, $\mathscr{M}^{2d}_{3(d,d+1)}$ *has* $\left\lfloor \frac{d}{2} \right\rfloor + 1$ *connected components parametrized by the sets* $\{\omega, \omega^{-1}\}$, *when* $\omega^d = 1$.

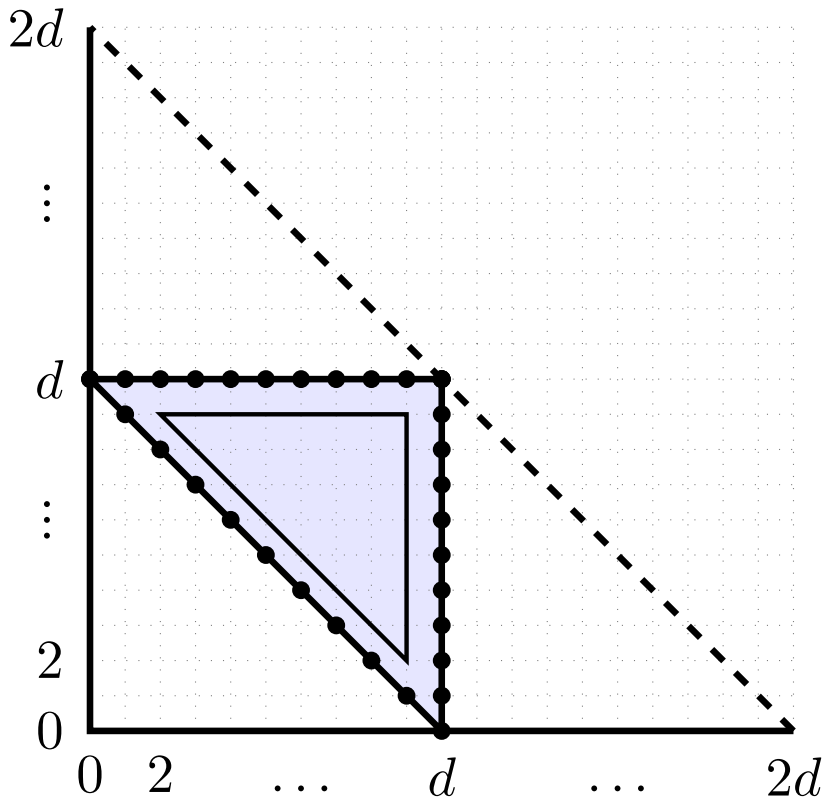


Realization space

Theorem

If $d \geq 3$, $\mathscr{M}^{2d}_{3(d,d+1)}$ *has* $\left\lfloor \frac{d}{2} \right\rfloor + 1$ *connected components parametrized by the sets* $\{\omega, \omega^{-1}\}$, *when* $\omega^d = 1$.

Proof.



Realization space

Theorem

If $d \geq 3$, $\mathscr{M}^{2d}_{3(d,d+1)}$ *has* $\left\lfloor \frac{d}{2} \right\rfloor + 1$ *connected components parametrized by the sets* $\{\omega, \omega^{-1}\}$, *when* $\omega^d = 1$.

$\mathscr{M}^{2d+3}_{3\langle uv((u+v)^d+v^{d+1})\rangle}$

- $\mathcal{D} \in \mathscr{M}^{2d+3}_{3\langle uv((u+v)^d+v^{d+1})\rangle}$



Realization space

Theorem

If $d \geq 3$, $\mathscr{M}^{2d}_{3(d,d+1)}$ *has* $\left\lfloor \frac{d}{2} \right\rfloor + 1$ *connected components parametrized by the sets* $\{\omega, \omega^{-1}\}$, *when* $\omega^d = 1$.

$\mathscr{M}^{2d+3}_{3\langle uv((u+v)^d+v^{d+1})\rangle}$

- $\mathcal{D} \in \mathscr{M}^{2d+3}_{3\langle uv((u+v)^d+v^{d+1})\rangle}$
- $\mathcal{L}$ line passing through two singular points



Realization space

Theorem

If $d \geq 3$, $\mathscr{M}^{2d}_{3(d,d+1)}$ *has* $\left\lfloor \frac{d}{2} \right\rfloor + 1$ *connected components parametrized by the sets* $\{\omega, \omega^{-1}\}$, *when* $\omega^d = 1$.

$\mathscr{M}^{2d+3}_{3\langle uv((u+v)^d+v^{d+1})\rangle}$

- $\mathcal{D} \in \mathscr{M}^{2d+3}_{3\langle uv((u+v)^d+v^{d+1})\rangle}$
- $\mathcal{L}$ line passing through two singular points
- $\mathcal{D} \cdot \mathcal{L} \geq 2(d+2) \Rightarrow \mathcal{L} \subset \mathcal{D}$



Realization space

Theorem

If $d \geq 3$, $\mathscr{M}^{2d}_{3(d,d+1)}$ *has* $\left\lfloor \frac{d}{2} \right\rfloor + 1$ *connected components parametrized by the sets* $\{\omega, \omega^{-1}\}$, *when* $\omega^d = 1$.

$\mathscr{M}^{2d+3}_{3\langle uv((u+v)^d+v^{d+1})\rangle}$

- $\mathcal{D} \in \mathscr{M}^{2d+3}_{3\langle uv((u+v)^d+v^{d+1})\rangle}$
- $\mathcal{L}$ line passing through two singular points
- $\mathcal{D} \cdot \mathcal{L} \geq 2(d+2) \Rightarrow \mathcal{L} \subset \mathcal{D}$
- $\mathcal{D} = \mathcal{C} \cup L_{QR} \cup L_{PR} \cup L_{PQ}$, $\mathcal{C} \in \mathscr{M}^{2d}_{3(d,d+1)}$



Realization space

Theorem

If $d \geq 3$, $\mathscr{M}^{2d}_{3(d,d+1)}$ *has* $\left\lfloor \frac{d}{2} \right\rfloor + 1$ *connected components parametrized by the sets* $\{\omega, \omega^{-1}\}$, *when* $\omega^d = 1$.

$\mathscr{M}^{2d+3}_{3\langle uv((u+v)^d+v^{d+1})\rangle}$

- $\mathcal{D} \in \mathscr{M}^{2d+3}_{3\langle uv((u+v)^d+v^{d+1})\rangle}$
- $\mathcal{L}$ line passing through two singular points
- $\mathcal{D} \cdot \mathcal{L} \geq 2(d+2) \Rightarrow \mathcal{L} \subset \mathcal{D}$
- $\mathcal{D} = \mathcal{C} \cup L_{QR} \cup L_{PR} \cup L_{PQ}$, $\mathcal{C} \in \mathscr{M}^{2d}_{3(d,d+1)}$

Theorem

If $d \geq 3$, $\mathscr{M}^{2d+3}_{3\langle uv((u+v)^d+v^{d+1})\rangle}$ *has* $\left\lfloor \frac{d}{2} \right\rfloor + 1$ *connected components parametrized by* $\{\{\omega, \omega^{-1}\} \mid \omega^d = 1\}$



Cremona transformation and algebraic computation of the linking invariant

After a Cremona transformation $[X : Y : Z] \mapsto [YZ : XZ : XY]$:
$XYZG_\omega(X, Y, Z) = 0$, smooth curve with three maximal flexes.



Cremona transformation and algebraic computation of the linking invariant

After a Cremona transformation $[X : Y : Z] \mapsto [YZ : XZ : XY]$:
$XYZG_\omega(X, Y, Z) = 0$, smooth curve with three maximal flexes.

- $\mathcal{S} = \{G_\omega = 0\}$, $\mathcal{L}_a = \{X = 0\}$, $\mathcal{L}_b = \{Y = 0\}$ and $\mathcal{L}_c = \{Z = 0\}$.



Cremona transformation and algebraic computation of the linking invariant

After a Cremona transformation $[X : Y : Z] \mapsto [YZ : XZ : XY]$:
$XYZG_\omega(X, Y, Z) = 0$, smooth curve with three maximal flexes.

- $\mathcal{S} = \{G_\omega = 0\}$, $\mathcal{L}_a = \{X = 0\}$, $\mathcal{L}_b = \{Y = 0\}$ and $\mathcal{L}_c = \{Z = 0\}$.
- $G_\omega(0, Y, Z) = (Y + Z)^d, G_\omega(X, 0, Z) = (X + Z)^d$



Cremona transformation and algebraic computation of the linking invariant

After a Cremona transformation $[X : Y : Z] \mapsto [YZ : XZ : XY]$:
$XYZG_\omega(X, Y, Z) = 0$, smooth curve with three maximal flexes.

- $\mathcal{S} = \{G_\omega = 0\}$, $\mathcal{L}_a = \{X = 0\}$, $\mathcal{L}_b = \{Y = 0\}$ and $\mathcal{L}_c = \{Z = 0\}$.
- $G_\omega(0, Y, Z) = (Y + Z)^d$, $G_\omega(X, 0, Z) = (X + Z)^d$
- $G_\omega(X, Y, 0) = (Y + \omega X)^d$



Cremona transformation and algebraic computation of the linking invariant

After a Cremona transformation $[X : Y : Z] \mapsto [YZ : XZ : XY]$:
$XYZG_\omega(X, Y, Z) = 0$, smooth curve with three maximal flexes.

- $\mathcal{S} = \{G_\omega = 0\}$, $\mathcal{L}_a = \{X = 0\}$, $\mathcal{L}_b = \{Y = 0\}$ and $\mathcal{L}_c = \{Z = 0\}$.
- $G_\omega(0, Y, Z) = (Y + Z)^d, G_\omega(X, 0, Z) = (X + Z)^d$
- $G_\omega(X, Y, 0) = (Y + \omega X)^d$
- $H_\omega = \{[X : Y : Z : T] \in \mathbb{P}^3 \mid T^d = G_\omega(X, Y, Z)\} \xrightarrow{\rho_\omega} \mathbb{P}^2$.



Cremona transformation and algebraic computation of the linking invariant

After a Cremona transformation $[X:Y:Z] \mapsto [YZ:XZ:XY]$:
$XYZG_\omega(X,Y,Z) = 0$, smooth curve with three maximal flexes.

- $\mathcal{S} = \{G_\omega = 0\}$, $\mathcal{L}_a = \{X = 0\}$, $\mathcal{L}_b = \{Y = 0\}$ and $\mathcal{L}_c = \{Z = 0\}$.
- $G_\omega(0,Y,Z) = (Y+Z)^d, G_\omega(X,0,Z) = (X+Z)^d$
- $G_\omega(X,Y,0) = (Y+\omega X)^d$
- $H_\omega = \{[X:Y:Z:T] \in \mathbb{P}^3 \mid T^d = G_\omega(X,Y,Z)\} \xrightarrow{\rho_\omega} \mathbb{P}^2$.
- $\rho_\omega^{-1}(\mathcal{L}_a) = \bigcup_{\zeta^d=1} \mathcal{L}_a^\zeta \quad , \mathcal{L}_a^\zeta = \{X = 0, T = \zeta(Y+Z)\}$



Cremona transformation and algebraic computation of the linking invariant

After a Cremona transformation $[X:Y:Z] \mapsto [YZ:XZ:XY]$:
$XYZG_\omega(X,Y,Z) = 0$, smooth curve with three maximal flexes.

- $\mathcal{S} = \{G_\omega = 0\}$, $\mathcal{L}_a = \{X = 0\}$, $\mathcal{L}_b = \{Y = 0\}$ and $\mathcal{L}_c = \{Z = 0\}$.
- $G_\omega(0,Y,Z) = (Y+Z)^d, G_\omega(X,0,Z) = (X+Z)^d$
- $G_\omega(X,Y,0) = (Y+\omega X)^d$
- $H_\omega = \{[X:Y:Z:T] \in \mathbb{P}^3 \mid T^d = G_\omega(X,Y,Z)\} \xrightarrow{\rho_\omega} \mathbb{P}^2$.
- $\rho_\omega^{-1}(\mathcal{L}_a) = \bigcup_{\zeta^d=1} \mathcal{L}_a^\zeta \quad , \mathcal{L}_a^\zeta = \{X = 0, T = \zeta(Y+Z)\}$
- $\rho_\omega^{-1}(\mathcal{L}_b) = \bigcup_{\zeta^d=1} \mathcal{L}_b^\zeta \quad , \mathcal{L}_b^\zeta = \{Y = 0, T = \zeta(X+Z)\}$



Cremona transformation and algebraic computation of the linking invariant

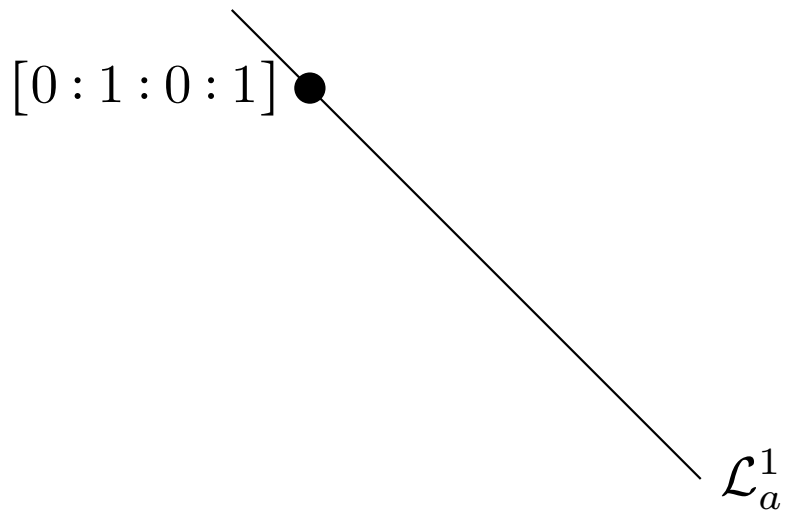
After a Cremona transformation $[X:Y:Z] \mapsto [YZ:XZ:XY]$:
$XYZG_\omega(X,Y,Z) = 0$, smooth curve with three maximal flexes.

- $\mathcal{S} = \{G_\omega = 0\}$, $\mathcal{L}_a = \{X = 0\}$, $\mathcal{L}_b = \{Y = 0\}$ and $\mathcal{L}_c = \{Z = 0\}$.
- $G_\omega(0,Y,Z) = (Y+Z)^d, G_\omega(X,0,Z) = (X+Z)^d$
- $G_\omega(X,Y,0) = (Y+\omega X)^d$
- $H_\omega = \{[X:Y:Z:T] \in \mathbb{P}^3 \mid T^d = G_\omega(X,Y,Z)\} \xrightarrow{\rho_\omega} \mathbb{P}^2$.
- $\rho_\omega^{-1}(\mathcal{L}_a) = \bigcup_{\zeta^d=1} \mathcal{L}_a^\zeta \quad , \mathcal{L}_a^\zeta = \{X = 0, T = \zeta(Y+Z)\}$
- $\rho_\omega^{-1}(\mathcal{L}_b) = \bigcup_{\zeta^d=1} \mathcal{L}_b^\zeta \quad , \mathcal{L}_b^\zeta = \{Y = 0, T = \zeta(X+Z)\}$
- $\rho_\omega^{-1}(\mathcal{L}_c) = \bigcup_{\zeta^d=1} \mathcal{L}_c^\zeta \quad , \mathcal{L}_c^\zeta = \{Z = 0, T = \zeta(X+\omega Y)\}$



Cremona transformation and algebraic computation of the linking invariant

After a Cremona transformation $[X : Y : Z] \mapsto [YZ : XZ : XY]$:
$XYZG_\omega(X, Y, Z) = 0$, smooth curve with three maximal flexes.

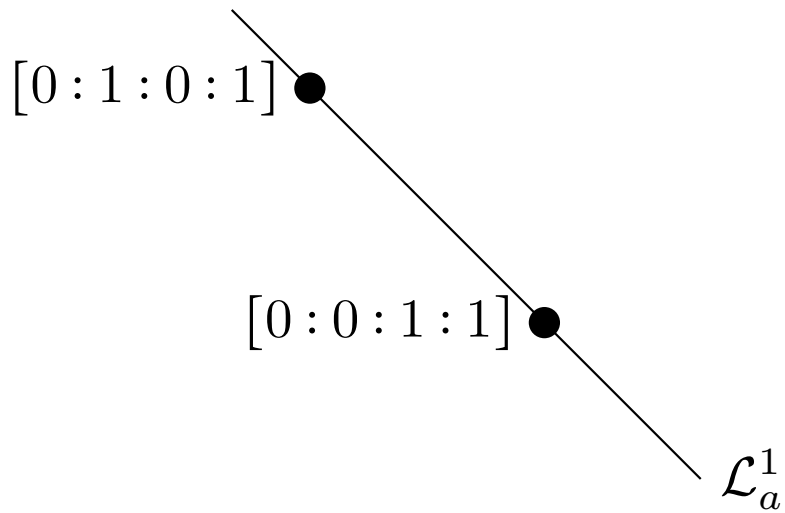


$\mathcal{L}_a^1 : X = 0, T = Y + Z$



Cremona transformation and algebraic computation of the linking invariant

After a Cremona transformation $[X : Y : Z] \mapsto [YZ : XZ : XY]$:
$XYZG_\omega(X, Y, Z) = 0$, smooth curve with three maximal flexes.

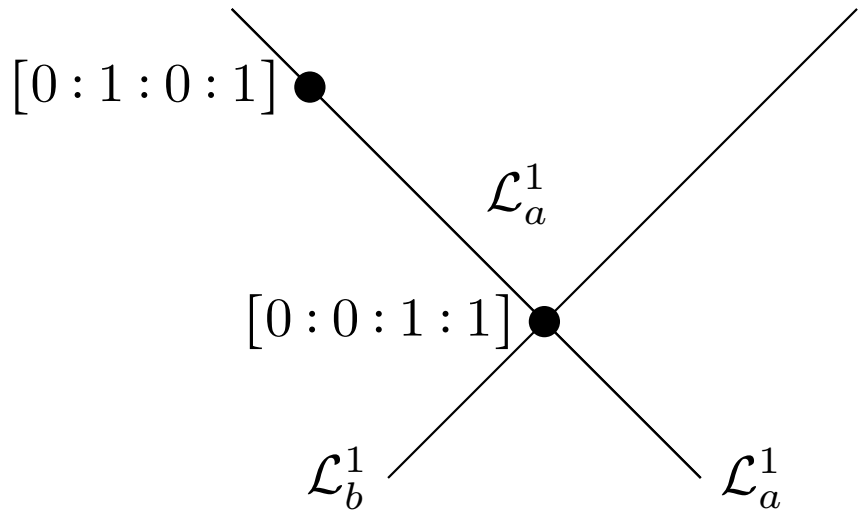


$\mathcal{L}_a^1 : X = 0, T = Y + Z$

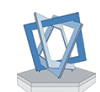


Cremona transformation and algebraic computation of the linking invariant

After a Cremona transformation $[X : Y : Z] \mapsto [YZ : XZ : XY]$:
$XYZG_\omega(X, Y, Z) = 0$, smooth curve with three maximal flexes.

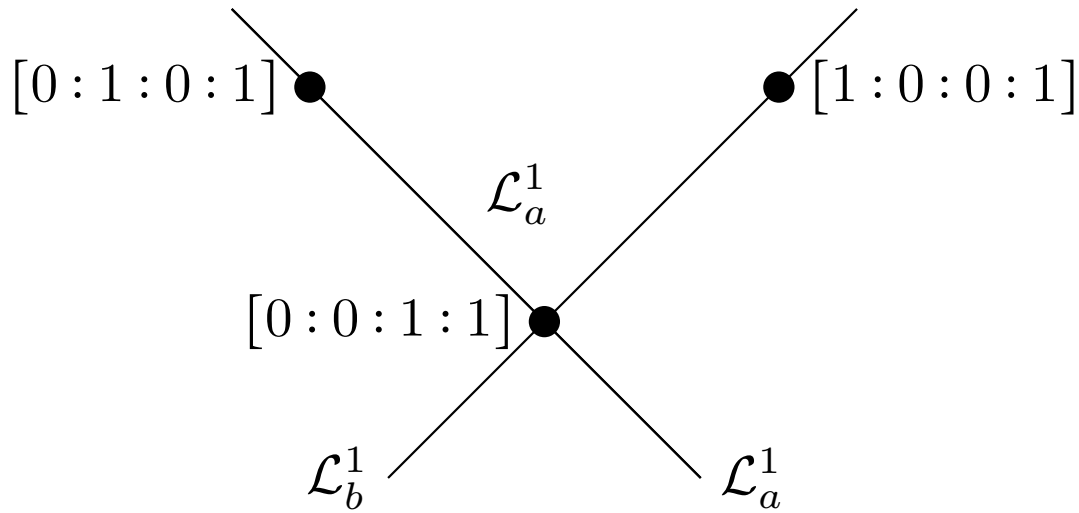


$\mathcal{L}_a^1 : X = 0, T = Y + Z$
$\mathcal{L}_b^1 : Y = 0, T = X + Z$

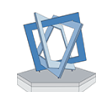


Cremona transformation and algebraic computation of the linking invariant

After a Cremona transformation $[X : Y : Z] \mapsto [YZ : XZ : XY]$:
$XYZG_\omega(X, Y, Z) = 0$, smooth curve with three maximal flexes.

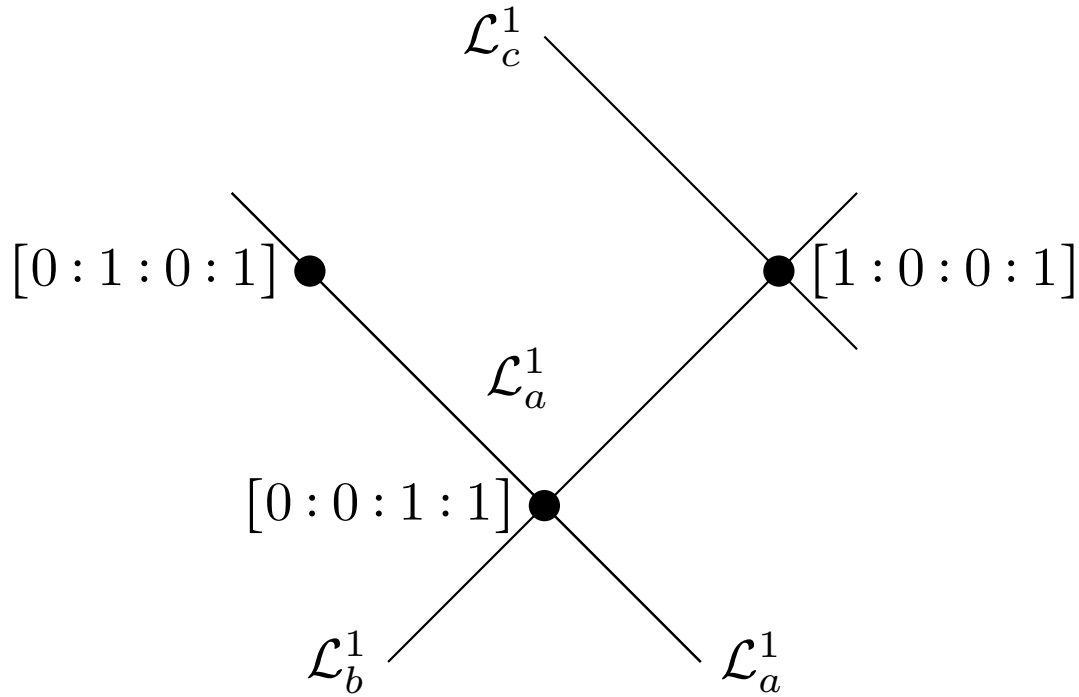


$\mathcal{L}_a^1 : X = 0, T = Y + Z$
$\mathcal{L}_b^1 : Y = 0, T = X + Z$



Cremona transformation and algebraic computation of the linking invariant

After a Cremona transformation $[X : Y : Z] \mapsto [YZ : XZ : XY]$:
$XYZG_\omega(X, Y, Z) = 0$, smooth curve with three maximal flexes.

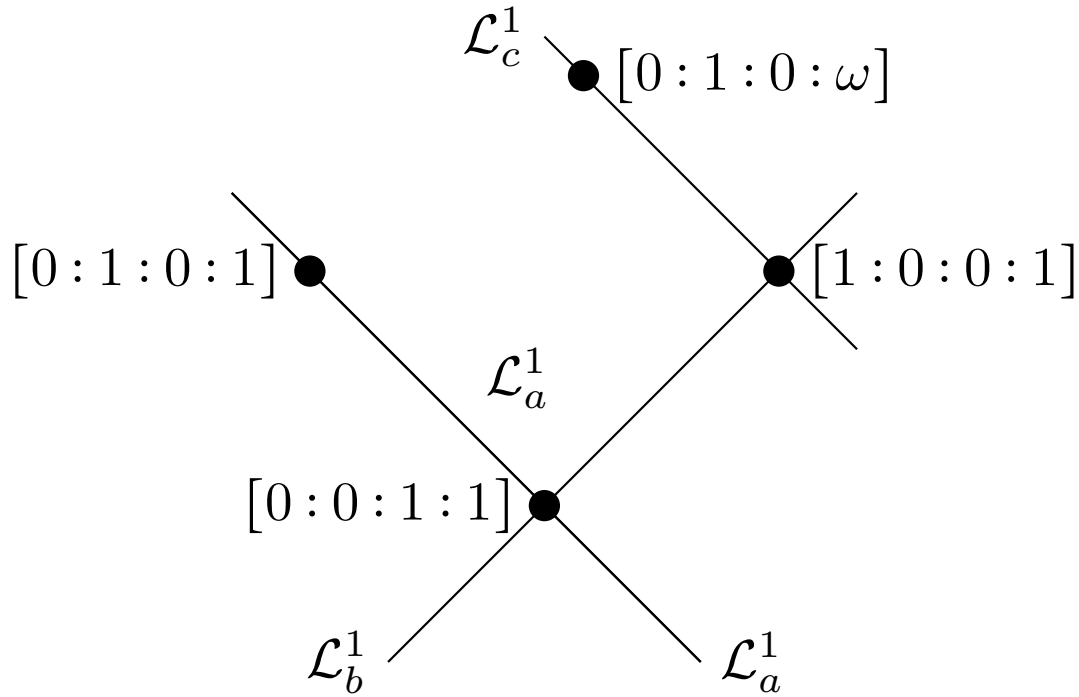


$$\mathcal{L}_a^1 : X = 0, T = Y + Z$$
$$\mathcal{L}_b^1 : Y = 0, T = X + Z$$
$$\mathcal{L}_c^1 : Z = 0, T = X + \omega Y$$



Cremona transformation and algebraic computation of the linking invariant

After a Cremona transformation $[X : Y : Z] \mapsto [YZ : XZ : XY]$:
$XYZG_\omega(X, Y, Z) = 0$, smooth curve with three maximal flexes.



$$\mathcal{L}_a^1 : X = 0, T = Y + Z$$
$$\mathcal{L}_b^1 : Y = 0, T = X + Z$$
$$\mathcal{L}_c^1 : Z = 0, T = X + \omega Y$$



Zariski tuple candidate

- Fundamental groups are almost always abelian (use Fermat curves and Kummer covers)
- The invariants are lost for $\mathscr{M}^{2d}_{3(d,d+1)}$.

