The Cohomology Algebra of a Plane Curve and Related Topics

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Definitions

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- Denote it by $\mathcal{E}^*_{\mathbb{P}^2}(\log \mathcal{C})$.
- $\mathcal{E}^*_{\mathbb{P}^2}(\log \mathcal{C})$ inherits a weight filtration W_*.

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Such a residue map will be denoted by $\operatorname{Res}^{[i]}$.

$$\begin{array}{ccccc} H^i(\mathbb{P}^2; W_i\mathcal{E}^*_{\mathbb{P}^2}(\log \mathcal{C})) & & & & \\ \wr\!\wr \ \text{Leray} & & & & \\ H^i(S; W_i\mathcal{E}^*_S(\log \bar{\mathcal{C}})) & \to & H^i(S; W_i/W_{i-1}) & \simeq & H^0(\bar{\mathcal{C}}^{[i]}) \\ \wr\!\wr \ \text{DeRham} & & & & \\ H^i(X) & & & & \end{array}$$

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In more generality:

$$H^i(\mathbb{P}^2; W_k\mathcal{E}^*_{\mathbb{P}^2}(\log \mathcal{C})) \xrightarrow{\operatorname{Res}^{[\mathbf{i},\mathbf{k}]}} H^{i-k}(\bar{\mathcal{C}}^{[k]}).$$

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2. *If* $\psi \in \mathcal{E}^2(\mathbb{P}^2)(\log \mathcal{C})$ *is such that* $\operatorname{Res}^{[2,2]} \psi = 0$ *and* $\operatorname{Res}^{[2,1]} \psi = 0$*, then* $\psi = 0$*.*

Example

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However, if $\psi = \varphi \frac{dx \wedge dy}{f}$, then

- $\varphi \in (x, y) \Rightarrow \psi \in \mathcal{E}_0^2(\log \mathcal{C})$.
- Moreover, if $\varphi \in (y) \Rightarrow \left(\mathrm{Res}^{[2,2]}\, \psi\right)_P = 0$ at all $P \in \bar{\mathcal{C}}^{[1]}$ infinitely near 0.

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- *Generators in degree 2:*

$$\begin{array}{cc} \psi_P^{\delta_1,\delta_2}, & P \in \mathcal{C}_i \cap \mathcal{C}_j, \delta_1 \in \Delta_P(\mathcal{C}_i), \delta_2 \in \Delta_P(\mathcal{C}_j) \\ \psi_\infty^{i,k_i}, & i = 1, \ldots, r, k_i = 1, \ldots, d_i - 1 \\ \eta^{i,s_i}, \bar{\eta}^{i,s_i}, & i = 1, \ldots, r, s_i = 1, \ldots, g_i. \end{array}$$

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- *Relations:*

$$\psi_P^{\delta_1,\delta_2} = -\psi_P^{\delta_2,\delta_1}$$
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for any $P \in \mathcal{C}_i \cap \mathcal{C}_j \cap \mathcal{C}_k$ and $\delta_1 \in \Delta_P(\mathcal{C}_i)$, $\delta_2 \in \Delta_P(\mathcal{C}_j)$, $\delta_3 \in \Delta_P(\mathcal{C}_k)$.

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- *Product:*

$$\sigma_i \wedge \sigma_j = \sum_{P \in \mathcal{C}_i \cap \mathcal{C}_j} \mu_P(\delta_1, \delta_2) \psi_P^{\delta_1,\delta_2} + d_i \sum_{k_j=1}^{d_j-1} \psi_\infty^{j,k_j} - d_j \sum_{k_i=1}^{d_i-1} \psi_\infty^{i,k_i}.$$

Remark

Note that from the given presentation one can deduce that $H^*(X)$ only depends on the following invariants of $\mathcal{C}$:

$$(\{1, ..., r\}, \mathcal{S} = \operatorname{Sing}\mathcal{C}, \{\Delta_P\}_{P\in\mathcal{S}}, \{\phi_P\}_{P\in\mathcal{S}}, \{\mu_P\}_{P\in\mathcal{S}})$$

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such an ordered set of invariants of $\mathcal{C}$ will be referred to as the *Weak Combinatorics of* $\mathcal{C}$.

Hence

Theorem

The cohomology algebra of X only depends on the weak combinatorics of $\mathcal{C}$ and the genera of its irreducible components.

Consider $\omega \in H^1(X)$.

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$$0 \to H^1(X) \xrightarrow{\bullet\wedge\omega} H^2(X) \to 0 \qquad (H^*(X), \wedge\omega)$$

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$$\mathcal{R}^i(X) := \{\omega \in H^1(X) \mid h^1(H^*(X), \wedge\omega) \geq i\}$$

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Remark

Note that for any graded algebra A^* one can analogously define the i-th Resonance Variety $\mathcal{R}^i(A)$ of A^*.

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$$A^1 := \sum_{i=1}^r \sigma_i \mathbb{C} \quad A^2 := \sum_{P \in \mathcal{S}} \frac{\bigwedge^2 A_P}{I_P},$$

where

$$A_P := \sum_{\delta \in \Delta_P} \psi_P^\delta \mathbb{C}$$

$$I_P := \langle \psi_P^{\delta_1} \wedge \psi_P^{\delta_2} + \psi_P^{\delta_2} \wedge \psi_P^{\delta_3} + \psi_P^{\delta_3} \wedge \psi_P^{\delta_1} \rangle_{\mathbb{C}}$$

and

$$\sigma_i \wedge \sigma_j := \sum_{P \in \mathcal{C}_i \cap \mathcal{C}_j} \mu_P(\delta_1, \delta_2) \psi_P^{\delta_1, \delta_2}$$

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Note that $A^2 \ncong H^2(X)$.

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Definition (J.Hilman, C.Sabbah, Esnault-Schechtman-Viehweg, M.Falk, D.Arapura)

The i-th *Characteristic Variety* $\operatorname{Char}^i(X)$ of a topological space X is

$$\{\rho \in \operatorname{Hom}(\pi_1(X), \mathbb{C}^*) \mid h^1(X; \mathbb{C}_\rho) \geq i\}$$

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Theorem (A.Libgober, D.Cohen-A.Suciu, E.Hironaka)

If X is the complement of a projective curve $\mathcal{R}^i(X)$ is the tangent cone of $\operatorname{Char}^i(X)$ at the origin $\mathbf{1}$.

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Corollary

The components of $\text{Char}^i(X)$ passing through $\mathbf{1}$ are combinatorially determined.

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Definition (P.Deligne-P.Griffiths-J.Morgan-D.Sullivan)

A differential space X is called *formal* if its algebra of global differential forms $(\mathcal{E}(X), d)$ is formal.

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$$\begin{array}{ccc} H^*(X) & \overset{e}{\to} & \mathcal{E}^*(\mathbb{P}^2)(\log \mathcal{C}) \\ [\sigma_i] & \mapsto & \sigma_i \\ [\psi_P^{\delta_1,\delta_2}] & \mapsto & \psi_P^{\delta_1,\delta_2} \\ [\psi_\infty^{i,k_i}] & \mapsto & \psi_\infty^{i,k_i} \\ [\eta^{i,s_i}] & \mapsto & \eta^{i,s_i} \end{array}$$

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Can we choose forms so that e is *well-defined*?

$$\psi_P^{\delta_1,\delta_2} + \psi_P^{\delta_2,\delta_3} + \psi_P^{\delta_3,\delta_1} = 0$$

Choose δ_P at each $P \in \mathcal{S}$, then

$$\psi_P^{\delta_1,\delta_2} = \psi_P^{\delta_P,\delta_2} - \psi_P^{\delta_P,\delta_1}$$

$$\sigma_i \wedge \sigma_j =$$

$$= \sum_{P \in \mathcal{C}_i \cap \mathcal{C}_j} \mu_P(\delta_i, \delta_j) \psi_P^{\delta_i, \delta_j} +$$

$$+ d_i \sum_{k_j=1}^{d_j - 1} \psi_\infty^{j, k_j} - d_j \sum_{k_i=1}^{d_i - 1} \psi_\infty^{i, k_i}.$$

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$$+ d_i \sum_{k_j=1}^{d_j-1} \psi_\infty^{j,k_j} - d_j \sum_{k_i=1}^{d_i-1} \psi_\infty^{i,k_i}.$$

Let $\mathcal{C}_i, \mathcal{C}_j, \mathcal{C}_k$ be such that:

- $d_i = d_j = d_k$
- $\mu_P(\delta_i, \mathcal{C}_j) = \mu_P(\delta_i, \mathcal{C}_k)$,
- $\mu_P(\delta_j, \mathcal{C}_i) = \mu_P(\delta_j, \mathcal{C}_k)$,
- $\mu_P(\delta_k, \mathcal{C}_i) = \mu_P(\delta_k, \mathcal{C}_j)$,

then

$$\sigma_i \wedge \sigma_j + \sigma_j \wedge \sigma_k + \sigma_k \wedge \sigma_i = 0 \tag{1}$$

Note that if $\mathcal{C}_k = \alpha \mathcal{C}_i + \beta \mathcal{C}_j$, then (1) is trivial.

Theorem (Max-Noether Fundamental Theorem (M.Noether,..., W.Fulton))

Let F, G, and H be three plane curves with no common components. If $H_P \in (F_P, G_P)$ at any $P \in V(F) \cap V(G)$, then there exist two forms $A, B \in \mathbb{C}[x, y, z]$ such that

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Remark

The conditions $H_P \in (F_P, G_P)$ at any $P \in V(F) \cap V(G)$ are commonly known as the *Noether Conditions*.

Definition

Three curves F, G, and H satisfying (▸) are said to belong to a *combinatorial pencil*.

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Theorem (-,M.A.Marco)

If F, G, and H belong to a primitive combinatorial pencil, then they belong to an algebraic pencil ($H = \alpha F + \beta G$).

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This proves the formality of X.

Open Problems

- Are there also *nice* combinatorial descriptions of $H^*(X)$ in higher dimensions?

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- Are the complements of hypersurfaces in the projective space formal?
- What about toric varieties, or weighted projective spaces?